Korteweg-deVries (1895):

$$\frac{1}{c_0} \eta_t + \eta_x + \frac{3}{2h} \eta \eta_x + \frac{h^2}{6} \eta_{xxx} = 0, \quad c_0 = \sqrt{gh} \quad \text{linear wave speed}$$

(Dimensional)

$\eta = \text{free surface elevation}$

Derived from water wave equations:

- inviscid, incompressible, irrotational
- ideal
- shallow

What does shallow mean? Non-dimensionalize:

$$X' = \frac{x}{L}, \quad T' = \frac{t}{T}, \quad \eta' = \frac{\eta}{\alpha} \Rightarrow \eta_x' = \frac{\eta_x}{L}, \quad \eta_t' = \frac{\eta_t}{T}$$

$$\alpha \left( \eta_{tt}' + \frac{6\alpha}{2\alpha} \eta_{xx}' \right) + \frac{3a^2}{2hL} \eta_\eta' \eta_{xx}' + \frac{ah}{6L^3} \eta_{xxx}' = 0$$

Let $\varepsilon = \frac{a}{h}, \quad \mu = \frac{h}{L}$, then

$$\varepsilon \mu \left( \eta_{tt}' + \eta_{xx}' \right) + \frac{3}{2} \varepsilon^2 \mu \eta'_x \eta''_x + \frac{1}{6} \varepsilon^3 \mu \eta^3_3' = 0$$

One more step: move to moving frame with wave speed 1 and consider scaled time

$$\begin{align*}
\tilde{\xi} &= X' - T', \quad \tilde{x} = \varepsilon \tilde{\xi}, \quad \tilde{t} = \varepsilon \tilde{t}' - \frac{1}{2} \tilde{t}'
\end{align*}$$

$$\varepsilon^2 \mu \eta_{tt}'' + \frac{3}{2} \varepsilon^2 \mu \eta'_x \eta''_x + \frac{1}{6} \varepsilon^3 \mu \eta^3_3' = 0$$
Asymptotic Principle of Minimal Balance:

Want \( \varepsilon^2 \mu = \varepsilon \mu^3 \Rightarrow \varepsilon = \mu^2 \ll 1 \)

or \( \frac{a}{h} \ll 1, \frac{h}{L} \ll 1 \)

small amplitude long (shallow) waves (weak nonlinearity) and \( \frac{a}{h} \sim \frac{h^2}{L^2} \)

\[
\eta'' + \frac{3}{2} \eta' \eta'' + \frac{1}{6} \eta''' \eta'' = 0 \quad \text{(KdV equation)}
\]

\rightarrow weak\ quadratic\ NL\ & weak\ disp.

\rightarrow also\ arises\ in\ many\ physical\ systems\ (WW,\ plasma,\ lattice\ dynamics,\ magma\ migration,\ optics,\ ...)

\rightarrow "Solvable" by IST (GGKM PRL (1967))

\rightarrow ushered in the field of nonlinear waves

\textbf{Solitary Waves} - exponentially localized traveling wave solns of\ dispersive NL wave eqtn.

Consider KdV \( u_t + uu_x + u_{xxx} = 0 \)

\rightarrow nl\ dispersion \( \rightarrow \) causes u to spread

\rightarrow nl\ steepening \( \rightarrow \) solitary wave

\rightarrow dispersion spreading

balance
Seek traveling wave solution:

\[ u = u(x - ct) = u(\xi), \quad \text{wth } \lim_{|\xi| \to \infty} u(\xi) = 0 \]

\[ -cu' + u'' = 0 \quad \text{OK} \]

\[ -cu'' = A \]

\[ -cu + u'' = 0 \quad \Rightarrow \quad u'' - cu = A = 0 \]

for \( s \gg 1 \), have \( |uu'| \ll |cu'|, \quad |u'''| \ll \) so

\[ u(\xi) \sim e^{-c|\xi|} \quad |\xi| \to \infty \]

\( \Rightarrow \) Necessary condition:

\[ c > 0 \]

otherwise have wave-type solns \( e^{i(kx - \omega t)} \)

Continuing:

\[ (-cu + \frac{1}{2}u^2 + u'' = A)u' \]

\[ \Rightarrow \quad -\frac{c}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{2}(u')^2 = Au + B \]

\[ \Rightarrow \quad \frac{1}{2}(u')^2 = -\frac{1}{6}u^3 + \frac{c}{2}u^2 + Au + B \]

\[ = G(u) \to \text{cubic poly} \]
Assume $G$ has 3 real roots:

$$u_1 \leq u_2 \leq u_3$$

for particular $A, B, C$.

Solitary wave is homoclinic orbit with $0 < u_0 \leq a < \max$.

We need a double root at $0$ in the phase plane.

Then we have:

\[
\begin{align*}
G(a) &= 0 \quad \Rightarrow \quad -\frac{1}{6} a^3 + \frac{c}{2} a^2 = 0 \\
G(0) &= 0 \quad \Rightarrow \quad B = 0 \\
G(0) &= 0 \quad \Rightarrow \quad A = 0
\end{align*}
\]

From these equations, we find:

- $G(a) = 0$ implies $a^2 (\frac{c}{2} - \frac{1}{6}) = 0$.
- $G(0) = 0$ implies $B = 0$.
- $G(0) = 0$ implies $A = 0$.

So, the solitary wave amplitude/speed relation is:

$$A = \frac{1}{3} a$$
Compute solution:

\[(A=B=0)\]:

\[(u')^2 = -\frac{1}{3} u^3 + c u^2 = u^2(c - \frac{1}{3} u)\]

Consider positive branch of \(u'\):

\[u' = +u\left(c - \frac{1}{3} u\right)^{1/2}\]

\[a = \frac{3c}{\sqrt{c} \sqrt{c - \frac{1}{3} a}}\]

\[\Rightarrow \int_{u(s)}^{\frac{a}{\sqrt{c} \sqrt{c - \frac{1}{3} a}}} \frac{d\tilde{u}}{\tilde{u} \sqrt{c - \frac{1}{3} \tilde{u}}} = \int_{s_0}^{s} d\xi = s_0 - s\]

Can check that:

\[\frac{d}{d\tilde{u}} \left( \frac{-2}{\sqrt{c}} \text{arcsech}\left(\sqrt{\frac{\tilde{u}}{3c}}\right) \right) = \frac{1}{\tilde{u} \sqrt{c - \frac{1}{3} \tilde{u}}}\]

\[\Rightarrow \frac{2}{\sqrt{c}} \text{arcsech}\left(\sqrt{\frac{\tilde{u}}{3c}}\right) = s_0 - s\]

\[\Rightarrow \sqrt{\frac{u}{3c}} = \text{sech}\left(\frac{\sqrt{c}}{2}(s_0 - s)\right) \Rightarrow u(x,t) = 3c \text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-s)\right)\]

KdV soliton solution

First observed in narrow Union canal near Edinburgh, Scotland in 1834, followed by horseback called "Great Primary Wave of Translation"

Measured \(a = 3c\) w/ expts in wave tank! B 1834 before solitary wave soln worked out by...
Boussinesq 1854 → said wave was linear

Boussinesq 1871, 72 → found solitary wave soln.

Rayleigh 1876

KdV 1895 → KdV eqtn., cnoidal wave solns

Kruskal, Zabusky 1965 → solitary waves interact elastically, behave like particles → soliton

\[ c = \sqrt{gh} \left( 1 + \alpha h \right) \frac{1}{2} \]