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Department of Mathematics

NC STATE UNIVERSITY

MA 305 Intro Elem Lin Algebra, final examination, May 16, 2000

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Your Name: ____________________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 4 problems, which are subdivided into 11 questions, where each question counts for the explicitly given number of points, adding to a total of **43 points**. Please write your answers in the spaces indicated, or below the questions (using the back of the sheets if necessary). You are allowed to consult **three** 8.5in $\times$ 11in sheets with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **120 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
Total	_____

Problem 1 (17 points) Please answer the following questions and **give a brief explanation for your answer**.

(a, 3 pts) True or false: for any matrix A with real entries, the number of linearly independent rows is equal to the number of linearly independent columns. Please explain.

(b, 3 pts) Please give a basis for the following orthogonal complement space: $\left(\mathrm{Span}_{\mathbb{R}}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right)^{\perp}$.

(c, 4 pts) Consider the following function on $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$: $\left\langle \begin{bmatrix}x_1\\x_2\end{bmatrix}, \begin{bmatrix}y_1\\y_2\end{bmatrix} \right\rangle = 1000x_1y_1 + \frac{1}{1000}x_2y_2$. Is this function an inner product in the vector space $\mathbb{R}^2$? Please explain.

(d, 4 pts) Consider the following function on $\mathbb{R}^2 \to \mathbb{R}^2$: $F\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+x_2\\x_1 \cdot x_2\end{bmatrix}$. Is F a linear transformation? Please explain.

(e, 3 pts) True or false: The Gram-Schmidt produces the **same set** of orthogonal basis vectors independently of the order in which the input basis vectors are processed. Please explain.

Problem 2: (9 pts)

(a, 4 pts) The matrix

$$\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

defines the rotation of any three-dimensional vector around one of the coordinate axes by a specific degree. Which axis and by what degree? Please explain.

(b, 5 pts) A plane $ax + by + c$ is to be best fitted to the values of $f(x,y)$ observed at (x,y):

x	0	1	-1	2	-2
y	0	0	1	1	2
$f(x,y)$	3	4	5	6	7

Please write down a 5×3 matrix A and a 5-dimensional vector b such that the solution to the corresponding least squares problem $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx b$ gives the coefficients of the optimal plane. You **do not** need to compute the solution.

Problem 3: (9 pts) Consider the QR-factorization of the 4×3 matrix A and a 4-dimensional vector b:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{R}, \quad b = \begin{bmatrix} -3 \\ -1 \\ -5 \\ 1 \end{bmatrix}.$$

(a, 6 pts) With the help of the QR-factorization, please solve the system of linear equations $Ax = b$ for the given matrix A and the given vector b. Please show all your work.

(b, 3 pts) Is the system $Ax = b$ from part a consistent? Please explain.

Problem 4: (8 pts) Consider the 2×2 matrix A and the corresponding system of linear differential equations.

$$A = \begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}, \qquad \begin{aligned} y_1' &= 5y_1 + 6y_2, \\ y_2' &= -3y_1 - 4y_2. \end{aligned}$$

(a) (a, 4 pts) Please compute the eigenvalues and corresponding eigenvectors for A.

(b) (b, 4 pts) Using the solution of part a, please give the general solution of the system of differential equations (with parameters that are determined by initial conditions).