Math 425 – Fall 2007 Name: SOLUTIONS

Test 1

Please, rigorously justify your answers and write clearly if you want credit for your work.

(1) (6 Pts) Are the following statements true or false? If you think that a statement is true, prove it, otherwise give a counterexample.

(a) Every bounded real sequence has a Cauchy subsequence. T

(b) If a real sequence (x_n) is convergent, then $(|x_n|)$ is also convergent. T

(c) If a real sequence $(|x_n|)$ is convergent, then (x_n) is also convergent. F

(d) If a real sequence $(|x_n|)$ is convergent to 0, then also (x_n) is convergent to 0. T

(a) By the Bolzano-Weierstrass Thm., every bounded real sequence has a convergent subsequence. Since every convergent sequence is a Cauchy sequence, then the statement is true.

(b) Since (x_n) is convergent, then, given any $\varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t.

$$|x_n - x| < \varepsilon, \quad \forall n \geq N(\varepsilon).$$

Then $\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon \quad \forall n \geq N(\varepsilon)$. Thus $(|x_n|)$ is also convergent.

(c) False Consider $(x_n) = (-1)^n$. (x_n) is divergent, but $(|x_n|) = (1)$ is convergent

(d) If $(|x_n|) \to 0$, then, given any $\varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t.

$$\big||x_n| - 0\big| = |x_n| < \varepsilon \quad \forall n \geq N(\varepsilon).$$

It follows that $|x_n - 0| = |x_n| < \varepsilon \quad \forall n \geq N(\varepsilon)$. Thus $(x_n) \to 0$.

(2) (2 Pts) Use the **definition of limit** to show that

$$\lim\left(\frac{n}{n^2+5}\right) = 0.$$

Need to show that, given any $\varepsilon > 0$, we can find $N(\varepsilon)$

s.t. $\left|\frac{n}{n^2+5} - 0\right| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$.

Observe that $\frac{1}{n} > \frac{1}{n+\frac{5}{n}} = \frac{n}{n^2+5}$, for $n \geq 1$

Thus, if $N = \frac{1}{\varepsilon}$, then $\frac{N}{N^2+5} < \frac{1}{N} = \varepsilon$

and $\left|\frac{n}{n^2+5} - 0\right| < \varepsilon \quad \forall\ n \geq N(\varepsilon) = \frac{1}{\varepsilon}$

(3) (2 Pts) Compute $\lim\left(\sqrt{(n^2+n)} - n\right)$. (State the properties and/or theorems you use when you deduce your result).

We have $\sqrt{(n^2+n)} - n = \frac{\left(\sqrt{n^2+n} - n\right)}{\sqrt{n^2+n} + n}\left(\sqrt{n^2+n} + n\right) =$

$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

Thus, using the Limit Theorems,

$$\lim_n\left(\sqrt{n^2+n} - n\right) = \frac{1}{\sqrt{\lim\left(1+\frac{1}{n}\right)}+1} = \frac{1}{2}$$

(4) (2 Pts) Let $a_1 = 1$, $a_{n+1} = \frac{1}{4}(2a_n + 5)$, $n \in \mathbb{N}$. Prove that (x_n) is monotone.

Observe that $a_2 = \frac{1}{4}(2+5) = \frac{7}{4} > a_1$

We will show that (a_n) is INCREASING

We use an induction argument

(1) $a_2 \geq a_1$ is true.

(2) Assume that $a_{n+1} \geq a_n$

(3) $a_{n+2} = \frac{1}{4}(2a_{n+1} + 5)$

$\geq \frac{1}{4}(2a_n + 5)$ by assumption in step 2)

$= a_{n+1}$ by definition.

Thus, $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$.