Fundamental Thm. of Calculus

THM (F.T.C. I form) Let $F : [a,b] \to \mathbb{R}$ s.t.

(a) F is continuous on $[a,b]$

(b) $F'(x) = f(x) \quad \forall x \in [a,b]$

(c) $f \in R[a,b]$

Then
$$\int_a^b f = F(b) - F(a)$$

Proof Fix $\varepsilon > 0$. Since $f \in R[a,b]$, then $\exists\, \delta_\varepsilon > 0$ s.t. if $\dot{P}$ is a tagged partition of $[a,b]$ and $\|\dot{P}\| < \delta_\varepsilon$, then $\left| S(f;\dot{P}) - \int_a^b f \right| < \varepsilon$.

Let $\dot{P} = \{ [x_{i-1}, x_i], t_i \}$

By the MVT, $F(x_i) - F(x_{i-1}) = \underbrace{F'(u_i)}_{= f(u_i)} (x_i - x_{i-1})$ for $i = 1, \dots, n$, where $u_i \in (x_{i-1}, x_i)$

Thus:
$$F(b) - F(a) = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) \right) = \sum_{i=1}^{n} f(u_i)\,(x_i - x_{i-1})$$

If we choose $t_i = u_i$ in the set $\dot{P}$, then
$$\left| (F(b) - F(a)) - \int_a^b f \right| < \varepsilon \quad \Longleftarrow \quad \text{whenever } \|\dot{P}\| < \delta_\varepsilon.$$

RK (b), can be replaced by $F'(x) = f(x) \quad \forall x \in [a,b] \setminus E$, where E is a finite set.

- If F is differentiable, then (a) is satisfied;
- Even if F is differentiable, condition (c) is not automatically satisfied.

Def If F exists satisfying the assumption of the theorem, then F is called the ANTIDERIVATIVE or PRIMITIVE of f.

Ex $F(x) = \dfrac{x^2}{2} \quad x \in [a,b]$

$f(x) = x \quad x \in [a,b]$. Then all assumptions of FTC are satisfied and
$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

Ex
$$A(x) = |x| \quad x \in [-10, 10]$$
$$A'(x) = \begin{cases} -1 & x \in [-10, 0) \\ 1 & x \in (0, 10] \end{cases} = \operatorname{sgn}(x) \quad x \in [-10,10] \setminus \{0\}$$

Notice that $\operatorname{sgn}(x) \in R[-10,10]$. Thus, by the FTC
$$\int_{-10}^{10} \operatorname{sgn} x\,dx = A(10) - A(-10) = 10 - 10 = 0$$

Ex Let $H(x) = 2\sqrt{x} \quad x \in [0,b]$

H is continuous on $[0,b]$ and $H'(x) = 1/\sqrt{x}$ for $x \in (0,b]$.

Since $h = H'$ is NOT bounded on $(0,b]$, then it does not belong to $R[0,b]$.

Thus, FTC does not apply. ($h = 1/\sqrt{x}$ is a generalized Riemann integrable on $(0,b]$).

Def If $f \in R[a,b]$, then the function

$$F(z) = \int_a^z f \quad , \quad z \in [a,b]$$

is the INDEFINITE INTEGRAL of f with base point a.

Thm The indefinite integral $F(z)$ is continuous on $[a,b]$.

In fact, if $|f(x)| \leq M \quad \forall x \in [a,b]$, then $|F(z) - F(w)| \leq M|z-w| \quad \forall z,w \in [a,b]$

Proof For $z,w \in [a,b]$, $w \leq z$, we have that

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

Thus $F(z) - F(w) = \int_w^z f$

Since $|f(x)| \leq M$ on $[a,b]$, then $\quad -M(z-w) \leq \int_w^z f \leq M(z-w)$

and $\quad |F(z) - F(w)| \leq M|z-w|$

Thm (FTC, Second Form) Let $f \in R[a,b]$ and f be continuous at $c \in [a,b]$.

Then $F(z) = \int_a^z f$ is differentiable at c and $F'(c) = f(c)$

Proof Let $c \in [a,b)$ and consider the right hand derivative of F at c.

Since f is continuous at c, given $\varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t.

if $c \leq x \leq c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \qquad (1)$$

Choose h satisfying $0 < h < \eta_\varepsilon$. Then

$$F(c+h) - F(c) = \int_c^{c+h} f$$

Using inequality (1), we have that

$$(f(c) - \varepsilon)h \leq F(c+h) - F(c) \leq (f(c) + \varepsilon)h$$

Thus: $$f(c) - \varepsilon \leq \frac{F(c+h) - F(c)}{h} \leq f(c) + \varepsilon$$

and $$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon$$

This shows that $$\lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$$

The left hand limit is computed in the same way.

Corollary If f is continuous on $[a,b]$, then $F(z) = \int_a^z f$ is differentiable on $[a,b]$ and $F'(x) =$

Ex $f(x) = \mathrm{sgn}(x)$ on $[-1,1]$

$f \in R[-1,1]$ and $F(x) = |x| - 1$ is its indef integral with basepoint -1

Since $F'(0)$ does not exist, then F is not the primitive of f on $[-1,1]$.

[Composition] Let $f \in R[a,b]$ with $f([a,b]) \subseteq [c,d]$ and $\varphi : [c,d] \to \mathbb{R}$ be continuous.

Then $\varphi \circ f \in R[a,b]$

Proof: If f is continuous at $u \in [a,b]$, then $\varphi \circ f$ is continuous at $u \in [a,b]$.

Since the set of points of discontinuity of f is a "set of measure zero" (null set)

so is the set of points of discontinuities of $\varphi \circ f$.

Thus $\varphi \circ f \in R[a,b]$

We have used

Lebesgue's Integrability Criterion: A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff it is continuous $\forall x \in [a,b] \setminus N$, where

N is a null set.

given any $\varepsilon > 0$

N is a null set $\Leftrightarrow$ $\exists \{(a_k, b_k)\}_{k=1}^{\infty}$ open intervals in $[a,b]$

s.t. $N \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ and $\sum_{k=1}^{\infty} |b_k - a_k| < \varepsilon$

This immediately implies that:

- Step functions $\in R[a,b]$
- monotone functions $\in R[a,b]$
- f on $[a,b]$ discontinuous on $\left\{\frac{1}{k} : k \in \mathbb{N}\right\}$ $\in R[a,b]$
- Dirichlet function $\notin R[a,b]$ since it is discontinuous $\forall x \in [a,b]$.

Corollary: Let $f \in R[a,b]$. Then $|f| \in R[a,b]$ and

$$\left| \int_a^b f \right| \le \int_a^b |f| \le M(b-a) \qquad \text{where } |f(x)| \le M \text{ on } [a,b]$$

Proof: By the composition theorem, let $\varphi(t) = |t|$. Then $\varphi \circ f \in R[a,b]$.

Since $-|f| \le f \le |f|$, and $|f(x)| \le M$, we have the inequalities.

Corollary: If $f, g \in R[a,b]$, then $fg \in R[a,b]$.

Proof: Use composition theorem with $\varphi(t) = t^2$. Then $f^2 = \varphi \circ f \in R[a,b]$.

Similarly, $g^2 \in R[a,b]$ and $(f+g)^2 \in R[a,b]$.

Thus $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in R[a,b]$.