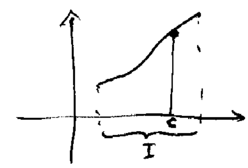
5.6 Monotone & Inverse Functions

$f: A \to \mathbb{R}$ Recall the def. of increasing / decreasing / monotone.

Notice If $f: A \to \mathbb{R}$ is increasing, then $(-f) = g$ is decreasing, and vice-versa.

Notice Monotonicity does not imply continuity.

Thm Let $f: I \to \mathbb{R}$ be increasing, and I be an interval. Suppose $c \in I$ is not an endpoint of I. Then



- $\lim\limits_{x \to c-} f = \sup\{f(x) : x \in I,\ x < c\}$
- $\lim\limits_{x \to c+} f = \inf\{f(x) : x \in I,\ x > c\}$

Proof We only prove the first statement.

If $x \in I$ and $x < c$, then $f(x) \le f(c)$. Hence $\{f(x) : x \in I,\ x < c\}$ is non empty since c is not end-point of I. Thus the sup described above exists. Call it L.

Let $\varepsilon > 0$.
$L - \varepsilon$ is not an upper bound of this set.
Hence $\exists\, y_\varepsilon \in I$, $y_\varepsilon < c$ s.t. $L - \varepsilon < f(y_\varepsilon) \le L$

Since f is increasing, if $\delta_\varepsilon = c - \varepsilon$, and $0 < c - y < \delta_\varepsilon$, then $y_\varepsilon < y < c$

so that $L - \varepsilon < f(y_\varepsilon) \le f(y) \le L$

Thus $|f(y) - L| < \varepsilon$ when $0 < c - y < \delta_\varepsilon$.

Corollary Let $f: I \to \mathbb{R}$ be increasing and I be an interval. Suppose c is not an endpoint of I. TFAE

(a) f is continuous at c

(b) $\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c+} f(x) = f(c)$

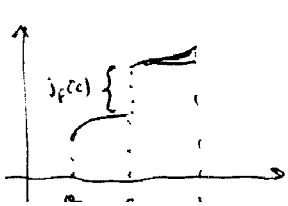
(c) $\sup\{f(x) : x \in I,\ x < c\} = f(c) = \inf\{f(x) : x \in I,\ x > c\}$

Consider $f: I \to \mathbb{R}$, increasing $\quad I = [a,b]$ or $I = (a,b]$, (a,b), $[a,b)$

$f(x)$ is continuous at $a \iff \lim\limits_{x \to a+} f(x) = f(a) \iff f(a) = \inf\{f(x) : x \in I,\ a < x\}$

If c is not and endpoint of I: ~~and~~ $\cancel{\lim\limits_{x \to c-} f(x) \ne \lim\limits_{x \to c+} f(x)}$,

we define $\quad j_f(c) = \lim\limits_{x \to c+} f(x) - \lim\limits_{x \to c-} f(x) \quad$ Jump of f at c



$j_f(a) = \lim\limits_{x \to a+} f - f(a) \quad$ Jump of f at a

$j_f(b) = f(b) - \lim\limits_{x \to b-} f \quad$ Jump of f at b

It follows from the definition that if $f: I \to \mathbb{R}$ is increasing, and $c \in I$, then f is continuous at $c \iff j_f(c) = 0$.

Thm Let $f: I \to \mathbb{R}$ be monotone on I, and I be an interval. Then the set of points where f is discontinuous is a countable set.

proof WLOG, suppose f increasing on I, and $I = [a,b]$.

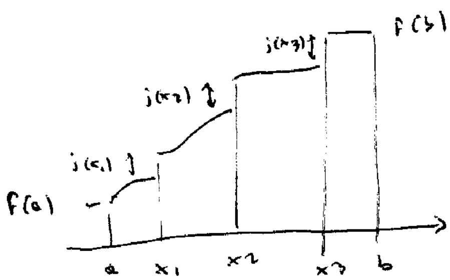
Let $D = \{c \in I : f \text{ is discontinuous at } c\}$

Then $D = \{c \in I : j_f(c) \neq 0\}$

Since f is increasing, then $j_f(c) = \lim_{x \to c+} f(x) - \lim_{x \to c-} f(x) \geq 0$

Also, for $a \leq x_1 \leq x_2 \ldots \leq x_n \leq b$, then

$$f(a) \leq f(a) + j_f(x_1) + \cdots + j_f(x_n) \leq f(b)$$



It follows that

$$j_f(x_1) + \cdots j_f(x_n) \leq f(b) - f(a)$$

If $j_f(x) \geq \dfrac{f(b) - f(a)}{k}$, then there are at most k discontinuity of this kind. This implies that there are at most a countable # of discontinuities x s.t. $j_f(x) > 0$.

Thus D must be a countable set

Corollary If $h(x)$ is monotone and $h(x+y) = h(x) + h(y)$ $\forall x, y \in \mathbb{R}$, then $h(x)$ is continuous on $\mathbb{R}$

proof Since $h(x)$ is monotone, then $h(x)$ has at most countably many discontinuities. Thus $h(x)$ is continuous at least at some point x_0.

It follows that $h(x)$ is continuous everywhere.

Inverse Function

Recall $f: I \to \mathbb{R}$ has an inverse (on $f(I)$) iff f is INJECTIVE (one-to-one),
[that is $x \neq y \Rightarrow f(x) \neq f(y)$]. ~~that is~~

Notice that if f is strictly monotone, then f is INJECTIVE.

<u>Thm</u> (<u>Continuous Inverse Thm</u>) Let $f: I \to \mathbb{R}$ be strictly monotone and continuous on the interval I. Then the inverse function g is strictly monotone and continuous on the interval $J = f(I)$.

<u>proof</u> WLOG assume f strictly increasing.

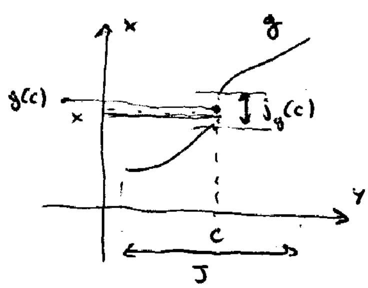
Since f is continuous, then $J = f(I)$ is an interval.

Since f is strictly increasing on I, then f is injective on I.

Thus the inverse function $g: f(I) \to \mathbb{R}$ exists.

We need to show that g is continuous and strictly increasing.

- Let $y_1, y_2 \in J$, $y_1 < y_2$. Then $y_1 = f(x_1)$, $y_2 = f(x_2)$ for some $x_1, x_2 \in I$.
  We must have $x_1 < x_2$, (otherwise $f(x_1) \geq f(x_2)$).
  Thus $g(y_1) = x_1 < x_2 = g(y_2)$ and g is strictly increasing.
- To show the continuity of g, suppose that g is discontinuous at $c \in J$. Then $\lim_{y \to c-} g(y) < \lim_{y \to c+} g(y)$.



Let x be such that $\lim_{y \to c-} g(y) < x < \lim_{y \to c+} g(y)$.
$x \neq g(c)$

Then $x \neq g(y)$ $\forall y \in J$ and this contradicts the fact that I is an interval. (since if $x \neq g(y)$, then $x \notin I$)

<u>APPLICATION</u>: Root Function

WANT TO SHOW THAT $f(x) = x^n$ has a strictly increasing and continuous inverse on $I = [0, \infty)$.

CASE (i) <u>n even</u> $f(x) = x^n$, $x \in I$, If $0 \leq x < y$, then $x^n < y^n$

Thus f is strictly increasing on I.

Since f is continuous, then $J = f(I)$ is an interval. (Claim: $J = [0, \infty)$)

In fact if $y > 0$ then $\exists k \in \mathbb{N}$ s.t. $0 \leq y < k$

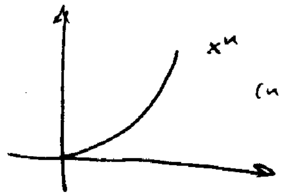
Since $\quad f(0)=0 \le y < k \le k^n = f(k)$

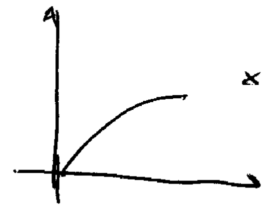
it follows by the Intermediate Value Thm. that $y \in J$.

Since y is arbitrary, then $J=[0,\infty)$.

Thus, by the Continuous Inverse Thm., the inverse func of f, say g, is strictly increasing and continuous on $J=[0,\infty)$.

We use the notation $g(x)=x^{\frac{1}{n}}$ or $g(x)=\sqrt[n]{x}$.

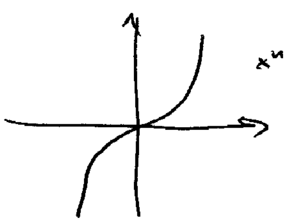


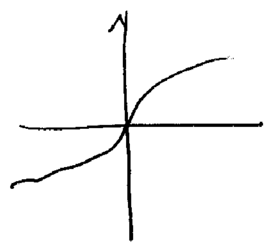


CASE (ii) n odd $\qquad F(x)=x^n$. $\qquad F$ is strictly increasing and continuous on $\mathbb{R}$.
$F(\mathbb{R})=\mathbb{R}$.

It follows that $F(x)$ has a strictly increasing and continuous inverse on $\mathbb{R}$.

$$G(x)=x^{1/n}$$





Def $\quad$ For $m,n\in\mathbb{N}$, $x\ge 0$, $\quad x^{\frac{m}{n}}=\left(x^{\frac{1}{n}}\right)^m$

For $m,n\in\mathbb{N}$, $x>0$ $\quad x^{-\frac{m}{n}}=\left(x^{\frac{1}{n}}\right)^{-m}$

One can show that $\quad x^{\frac{m}{n}}=(x^m)^{\frac{1}{n}}$