(2.5) Intervals

Def For $a,b \in \mathbb{R}$, with $a<b$, the OPEN INTERVAL (a,b) is
$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$
a, b are the ENPOINTS of the interval

$$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$$ is a CLOSED INTERVAL

$[a,b)$, $(a,b]$ are half-open intervals

$b-a$ is the LENGTH of the interval

$[a,\infty)$, $(-\infty, b]$ are infinite open/closed intervals.

Thm If $S \subset \mathbb{R}$ contains at least 2 points, and satisfies the property that, if $x<y$ for $x,y \in S$, then $[x,y]$ ~~⊆~~ then S is an interval.

Proof Need to discuss several cases depending on S bdd / unbounded

CASE 1 [S is bdd] let $a = \inf S$, $b = \sup S$. Then ~~$[a,b]$~~ $S \subseteq [a$

Need to show that $(a,b) \subseteq S$. This will prove that S is an interval.

If $a < z < b$, then z is not a lower bdd of S, and $\exists x \in$ S.t. $x<z$. Also z is not an upper bdd of S, and $\exists y \in S$ s. Thus $z \in [x,y]$. By hyp it follows that $z \in S$. Since z is arbitrary, ~~it follows~~ on (a,b), it follows that $(a,b) \subseteq S$.

Observe that, if $a,b \in S$, then $S = [a,b]$. If $a \notin S$ or $b \notin S$ th $S = (a,b)$ or $[a,b)$ or $(a,b]$.

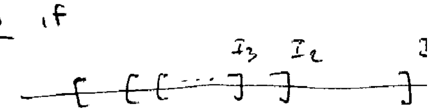
CASE 2 [S is bdd above but not below]. let $b = \sup S$. Then $S \subseteq (-\infty$ and we have to show that $(-\infty, b) \subseteq S$

If $z < b$, then $\exists x,y \in S$ s.t. $z \in [x,y] \subseteq S$. Thus, since z is arbitrary on $(-\infty,b)$, it follows that $(-\infty, b) \subset S$.

If $b \in S$, then $S = (-\infty, b]$, otherwise $(-\infty,b) = S$.

SIMILARLY for S bdd below and not above and for S unbdd above and below.

Def A sequence of intervals I_n, $n \in \mathbb{N}$, is NESTED if
$$I_1 \supseteq I_2 \supseteq I_3 \cdots \supseteq I_n$$



Ex $I_n = [0, \frac{1}{n}]$, $n \in \mathbb{N}$.

Is the intersection of nested intervals non-empty?

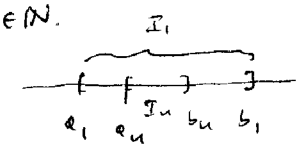
Ex $I_n = [0, \frac{1}{n}]$ $\quad 0 \in I_n \ \forall n.$

If $J_n = (0, \frac{1}{n})$, then, for each $x > 0$, there is $m \in \mathbb{N}$ s.t. $\frac{1}{m} < x$

and thus $x \notin J_m$.

Thus $\bigcap J_n$ is empty.

Thm (Nested Intervals Property) If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a seq. of closed bounded intervals, then $\exists\ \xi \in I_n$ s.t. $\xi \in I_n \ \forall n \in \mathbb{N}$.



proof Since $I_n \subseteq I_{n-1}$, then $I_n \subseteq I_1 \ \forall n$,

so that $a_n \le b_1$. Hence $\{a_n : n \in \mathbb{N}\}$ is a bdd set.

Let $\xi = \sup\{a_n : n \in \mathbb{N}\}$. We know that $a_n \le \xi \ \forall n$.

We will show that $\xi \le b_n \ \forall n$, since $b_n \ge a_k \ \forall k$.

In fact, (1) if $n \le k$, then $I_k \subseteq I_n$ and $a_k \le b_k \le b_n$

(2) if $k < n$, then $I_k \supseteq I_n$ and $a_k \le a_n \le b_n$

Thus $a_k \le b_n \ \forall k$, so that b_n is an upper bound for $\{a_k : k \in \mathbb{N}\}$

Hence $\xi \le b_n \ \forall n$, Since $a_n \le \xi \le b_n \ \forall n$, then $\xi \in I_n \ \forall n$

Thm If $J_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed bdd intervals s.t.

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

then the number $\xi \in I_n \ \forall n$ is unique.

Using the Nested Intervals Property, we will show that $\mathbb{R}$ is uncountable.

Thm $\mathbb{R}$ is NOT COUNTABLE

proof We will show that $[0,1]$ is uncountable.

We will argue by contradiction. Assume that $I = [0,1]$ is countable,

and so we can write $I = \{x_1, x_2 \dots\}$.

Choose I_1, closed set, with $I_1 \subset I$ s.t. $x_1 \notin I_1$, next choose $I_2 \subset I$, s.t. $x_2 \notin$

and so on. This way, we can build a sequence of nonempty,

closed nested intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \dots$$

with $I_n \subseteq I$, $x_n \notin I_n \ \forall n$.

The Nested Intervals Prop. implies that $\exists\ \xi \in I$ s.t. $\xi \in I_n \ \forall n$.

Thus $\xi \neq x_n \ \forall n$ so that the enumeration of I is not complete –

This is a contradiction. Hence I must be uncountable.