② Real Numbers $\mathbb{R}$

2.1 Algebraic Properties

On the set $\mathbb{R}$, we define ADDITION and MULTIPLICATION with their properties (see textbook)

Order Properties

There is a subset of $\mathbb{R}$, say $\mathbb{P}$, of POSITIVE ~~and~~ REAL NUMBERS satisfying

(i) $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.
(ii) $a, b \in \mathbb{P}$, then $a.b \in \mathbb{P}$
(iii) If $a \in \mathbb{R}$, then exactly one of the following holds

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}$$

If $a \in \mathbb{P}$, then a is a POSITIVE real number [RK 0 is NOT a Positive Real]

It follows that one can ORDER real numbers, by using inequalities

(see textbook)

RK No smallest positive real number exists. If $a > 0$, then $\frac{1}{2}a > 0$.
Thus we can always find a positive real smaller than any given one.

Arithmetic-Geometric Mean Inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \qquad \text{For all } a, b \in \mathbb{P}$$

Equality occurs iff $a = b$

proof ~~$0 < (a-b)^2 = a^2 + b^2 - 2ab$~~ $\quad 0 < (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$

~~$\Rightarrow a^2 + b^2 > 2ab \Rightarrow a$~~ $\quad \Rightarrow \quad a + b > 2\sqrt{ab}$

$$\Rightarrow \quad \frac{1}{2}(a+b) > \sqrt{ab}$$

Equality is obtained by observing that if

$$\frac{1}{2}(a+b) = \sqrt{ab} \Rightarrow (a+b)^2 = 4ab$$
$$\Rightarrow a^2 - 2ab + b^2 = 0$$
$$\Rightarrow (a-b)^2 = 0$$
$$\Rightarrow a = b$$

2.2 Absolute Value

Def Let $a \in \mathbb{R}$. The ABSOLUTE VALUE of a, denoted by $|a|$ is

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Ex. If $a = 3$, then $|3| = 3$

If $a = -\sqrt{2}$, then $|-\sqrt{2}| = \sqrt{2}$

Properties

- $|ab| = |a||b|$
- $a^2 = |a|^2$
- For $c \geq 0$, then $|a| \leq c \Leftrightarrow -c \leq a \leq c$

  In fact $|a| \leq c$ ~~implies that~~ $\Leftarrow$ $a \leq c$ and $-a \leq c$

  ~~then~~ $\Leftrightarrow$ $a \geq -c$, $a \leq c$

  Combining these observation: $-c \leq a \leq c$

- $-|a| \leq a \leq |a|$
- (Triangle Inequality) $|a+b| \leq |a| + |b|$

  In fact $-|a| \leq a \leq |a|$, $-|b| \leq b \leq |b|$

  then $-(|a|+|b|) \leq a+b \leq |a|+|b|$

  Thus $|a+b| \leq |a| + |b|$

- $|a-b| \leq |a| + |b|$
- $||a|-|b|| \leq |a-b|$

  In fact $|a| = |a-b+b| \leq |a-b| + |b|$

  $\Rightarrow |a-b| \geq |a| - |b|$

  $|b| = |b+a-a| \leq |b-a| + |a|$

  $\Rightarrow |a-b| \geq |b| - |a|$

  Thus $-|a-b| \leq |a|-|b| \leq |a-b|$

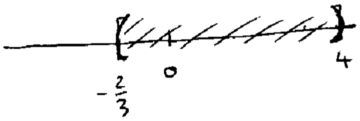
  $\Rightarrow ||a|-|b|| \leq |a-b|$

Ex Compute the set of the solutions of

$$|3x-5| < 7$$

Solution $-7 < 3x-5 < 7$

$$\Rightarrow \begin{cases} 3x-5 < 7 \Rightarrow 3x < 12 \Rightarrow x < 4 \\ 3x-5 > -7 \Rightarrow 3x > -2 \Rightarrow x > -\frac{2}{3} \end{cases}$$

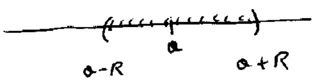


In general:

Given $a \in \mathbb{R}$, $R > 0$ $V_R(a) = \{x \in \mathbb{R} : |x-a| < R\}$

is the R-neighborhood of a given by

$$V_R(a) = (a-R, a+R)$$



Ex let $f(x) = \frac{2x^2+3x+1}{2x-1}$ $x \in (2,3)$

Find M s.t. $|f(x)| \le$ $\forall x \in (2,3)$

Solution

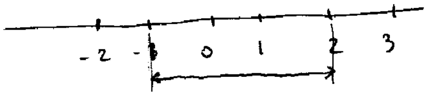
Observe: $|2x^2+3x+1| \le 2|x|^2 + 3|x| + 1 \le 2 \cdot 9 + 3 \cdot 3 + 1 = 28$

$$|2x-1| \ge 2|x|-1 \ge 3$$

Thus $|f(x)| \le \frac{28}{3}$

The absolute value is used to measure the distance between $a, b \in \mathbb{R}$.

$$d(a,b) = |a-b|$$



$$d(-1,2) = |-1-2| = 3$$

For $a \in \mathbb{R}$, $d(x,a) = |x-a|$ measure the distance between $x \in \mathbb{R}$

2 Completeness Property of ℝ.

ℝ is a COMPLETE ORDERED FIELD ↖ algebraic prop.
↑ ordered prop

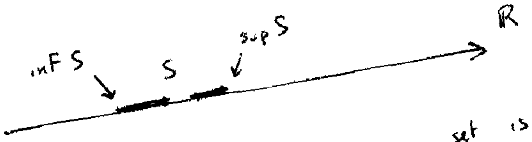
Observe that ℚ has also order and algebraic prop. However ℝ has the additional prop. that allows us to define limiting operations.

Def Let $S \subset \mathbb{R}$, $S \neq \emptyset$

(a) S is BOUNDED ABOVE if $\exists\, u \in \mathbb{R} : s \leq u \quad \forall s \in S$. u is an UPPER BOUND of S

(b) S is BOUNDED BELOW if $\exists\, w \in \mathbb{R} : w \leq s \quad \forall s \in S$. w is a LOWER BOUND of S

(c) S is BOUNDED if it is both bounded above and below. Otherwise, S is UNBOUNDED.



Ex $S = \{x : x > 0\}$ This set is bounded below. 0 is a LOWER BOUND. (But also −1, −2 ...) S is UNBOUNDED.

Def Let $S \subset \mathbb{R}$, $S \neq \emptyset$

(a) If S is bounded above, then $u = \sup S$ (supremum of S / least upper bound) if
- u is an upper bound of S
- $u \leq v$ for any upper bound v of S

(b) If S is bounded below, then $w = \inf S$ (infimum of S / greatest lower bound) if
- w is a lower bound of S
- $t \leq w$ for any lower bound t of S

Rk Not every set S has a sup or inf. If an inf or sup exists, they are unique.

Ex $S = \{x : x > 0\}$. This set has no $\sup S$ (in fact, has no upper bound)
- $\inf S = 0$ (but $0 \notin S$)

Ex $S = \{x : x \geq 0\}$
$\inf S = 0$, 0

Rk For a set S there are 4 possibilities
(1) S has both sup and inf
(2) S has sup but no inf
· S has inf but no sup
no inf

Properties let $S \subseteq \mathbb{R}$ be nonempty

- $u = \sup S \iff$ (1) $s \le u \quad \forall u \in S$
  (2) if $v < u$, then $\exists\, s' \in S : v < s'$

∘ let u be an upper bound of S. then

$u = \sup S \iff$ For each $\varepsilon > 0$, $\exists \; s_\varepsilon \in S : \; u - \varepsilon < s_\varepsilon$

proof $(\Rightarrow)$ let $u = \sup S$ and $\varepsilon > 0$.
Since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound
Then $\exists \; s_\varepsilon \in S : \quad s_\varepsilon > u - \varepsilon$. Thus $u - \varepsilon < s_\varepsilon$

$(\Leftarrow)$ let u be an upper bound of S and $v < u$.
let $\varepsilon = u - v$. Then $\exists \; s_\varepsilon$ s.t. $v = u - \varepsilon < s_\varepsilon$
Thus v is not a sup. Thus $u = \sup S$.

- If $S \neq \emptyset$ has finitely many elements, then $\sup S = \max S$ (largest elt)
  and $\inf S = \min S$ (smallest elt).

COMPLETENESS PROPERTY of $\mathbb{R}$ Every nonempty set of $\mathbb{R}$ that has an upper bound, also has a supremum in $\mathbb{R}$. (Same for the inf).

RK ∘ This property does not follow from the algebraic prop. of $\mathbb{R}$.
- This property is NOT true for $\mathbb{Q}$.

2.4 Applications of the Supremum Property

let $S \subseteq \mathbb{R}$, $S \neq \emptyset$, and $a \in \mathbb{R}$.

Define $a + S = \{a + s : s \in S\}$

claim: $\sup(a + S) = a + \sup S$.

proof let $u = \sup S$. Then $x \le u \quad \forall x \in S$
Then $a + x \le a + u \quad \forall x \in S$
This shows that $\sup(a+S) \le a + \sup S$

Def
Let $f : D \to \mathbb{R}$. f is BOUNDED ABOVE if the set $f(D) = \{f(x) : x \in D\}$ is bdd above
f is BOUNDED BELOW 〃 〃 bdd below
f is BOUNDED if it is bdd above and below

Ex $f(x) = x^2 \qquad -1 \le x \le 1$

$f(D) = \{x^2 : -1 \le x \le 1\} \qquad f(x) \le 1 \quad \forall x \in D$
$f(x) \ge 0 \quad \forall x \in D$

If $f(x) \le g(y) \quad \forall x, y \in D$, then $\sup f(D) \le \inf g(D)$.

Archimedean Property If $x \in \mathbb{R}$, then $\exists\, n_x \in \mathbb{N}$ s.t. $x < n_x$

This shows that the NATURAL NUMBERS are not bdd

It follows that:

(proofs in book)

- $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$
- if $t > 0$, $\exists\, n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$
- if $y > 0$, $\exists\, n_y \in \mathbb{N}$ s.t. $n_y - 1 \le y < n_y$

One of the consequence of density properties is the existence of irrational numbers. This was discovered by Greek math. already.
We have shown that $\mathbb{Q}$ is denumerable. However $\mathbb{R}$ is uncountable.
However, $\mathbb{Q}$ is dense in $\mathbb{R}$, in the sense that given any 2 real numbers, there is a real number between them:

Thm (Density Thm) If $x, y \in \mathbb{R}$ with $x < y$, then $\exists\, r \in \mathbb{Q}$: $x < r < y$

proof Let $x > 0$ (the argument is the same for $x < 0$).

Then $y - x > 0$ and, thus, $\exists\, n \in \mathbb{N}$ s.t. $y - x > \frac{1}{n}$.

This implies that $ny - nx > 1 \iff nx + 1 < ny$

~~It follows that~~ ~~$m \le nx + 1 < ny$ and, thus~~,

Take $m \in \mathbb{N}$ s.t. $m - 1 \le nx < m$

~~$m \le ny$~~

then $m \le nx + 1 < ny \implies m < ny$

Thus $nx < m < ny$

Thus $x < \frac{m}{n} < y$. Set $r = \frac{m}{n}$ ▨

Corollary If $x, y \in \mathbb{R}$, with $x < y$, then $\exists\, z \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $x < z < y$

proof Use thm above with

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}, \text{ then } x < \underset{\substack{\uparrow \\ \sqrt{2}\, r \,\in\, \mathbb{R}\setminus\mathbb{Q}}}{\sqrt{2}\, r} < y$$