Math 242 Sec. 8 – Fall 2007 Name: SOLUTION

TEST #1

No calculators, books or notes allowed. Please, justify your answers and write clearly if you want credit for your work.

(1) [3 Pts.] Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where $P(1,1,1)$, $Q(2,3,-1)$, $R(1,2,2)$, $S(6,-2,2)$.

$\vec{PQ} = (1,2,-2)$, $\vec{PR} = (0,1,1)$, $\vec{PS} = (5,-3,1)$ [1 Pt]

$$V = \vec{PQ} \cdot \vec{PR} \times \vec{PS} \quad [1\text{ Pt}]$$

$$= \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 5 & -3 & 1 \end{vmatrix} = 1\cdot 4 - 2(-5) - 2(-5) = 24 \quad [1\text{ Pt}]$$

(2) [7 Pts.] Let $\mathbf{r}(t) = (t^2, 2t, \ln t)$.
(a) Find a parametric equation of the tangent line to $\mathbf{r}(t)$ at $P(1,2,0)$.
(b) Find the curvature of $\mathbf{r}(t)$, as a function of t (recall: $k(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$).

(a) $\underline{r}'(t) = \left(2t, 2, \frac{1}{t}\right)$ $\quad \underline{r}'(1) = (2,2,1)$ [1 Pt]

$P(1,2,0) = \underline{r}(t=1)$

TANGENT LINE $\boxed{s(t) = (1+2t,\ 2+2t,\ t)}$ [2 Pt]

(b) $\underline{r}''(t) = (2, 0, -1/t^2)$ [1 Pt]

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} i & j & k \\ 2t & 2 & 1/t \\ 2 & 0 & -1/t^2 \end{vmatrix} = \left(-\frac{2}{t^2}, \frac{4}{t}, -4\right) \quad [1\text{ Pt}]$$

$$|\underline{r}'(t) \times \underline{r}''(t)| = \sqrt{\frac{4}{t^4} + \frac{16}{t^2} + 16} = \left(\frac{2}{t^2} + 4\right) \quad [1\text{ Pt}]$$

$$|\underline{r}'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \left(2t + \frac{1}{t}\right)$$

$$\boxed{k(t) = \frac{\sqrt{\frac{4}{t^4} + \frac{16}{t^2} + 16}}{\left(4t^2 + 4 + \frac{1}{t^2}\right)^{3/2}} = \frac{2}{\sqrt{t(2t^2+1)}}} \quad [1\text{ Pt}]$$

(3) [2 Pts.] Find the equation of the sphere passing through the point $(2, 1, 1)$ and with center $(1, 2, -1)$.

$R = \text{dist}\left[(1,2,-1), (2,1,1)\right] = \sqrt{1+1+4} = \sqrt{6}$ [1Pt]

SPHERE : $(x-1)^2 + (y-2)^2 + (z+1)^2 = 6$ [1Pt]

(4) [8 Pts.] (a) Find a parametric equation of the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

(b) Find the equation of the plane containing the point $P(1, 0, 1)$, and the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

(a) Intersection $\begin{cases} x+y-z=2 \\ 2x-y+3z=1 \end{cases} \Rightarrow z = x+y-2$

$\Rightarrow 2x - y + 3(x+y-2) = 1$

$5x + 2y = 7 \Rightarrow \begin{cases} x = \frac{7}{5} - \frac{2}{5}y \\ z = \frac{7}{5} - \frac{2}{5}y + y - 2 = -\frac{3}{5} + \frac{3}{5}y \end{cases}$

Set $y = 5t$

[2Pt]

Line : $\boxed{r(t) = \left(\frac{7}{5} - 2t, 5t, -\frac{3}{5} + 3t\right)}$ [2Pt]

Notice $r\left(\frac{1}{5}\right) = (1,1,0) = Q$

(b) PLANE must contain vector $v = (-2, 5, 3)$

vector $\vec{PQ} = (0, 1, -1)$ [2Pt]

NORMAL VECTOR is $n = \vec{PQ} \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ -2 & 5 & 3 \end{vmatrix} = (8, -2, 2)$ [1Pt]

Eq. of plane:

$8(x-1) + 2y + 2(z-1) = 0$

$\boxed{4x + y + z = 5}$ [1Pt]