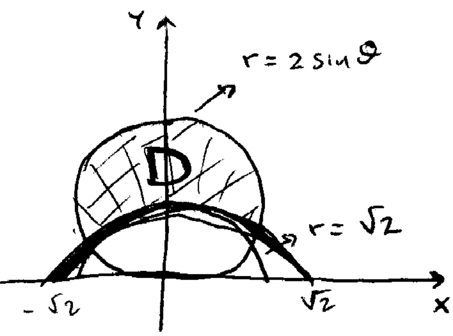
Math 242 Sec. 7 – Fall 2005 Name: SOLUTION

TEST #3

One page of notes allowed. You can use a hand calculator. Please, justify your answers and write clearly if you want credit for your work. Each problem is worth 4 points

(1) A lamina occupies the region **inside** the circle $x^2 + y^2 = 2y$ but **outside** the circle $x^2 + y^2 = 2$. Find the mass if the density is $\rho(x, y) = 1$.



$x^2+y^2=2y \iff r = 2\sin\vartheta$ (1pt)

$x^2+y^2=2 \iff r=\sqrt{2}$

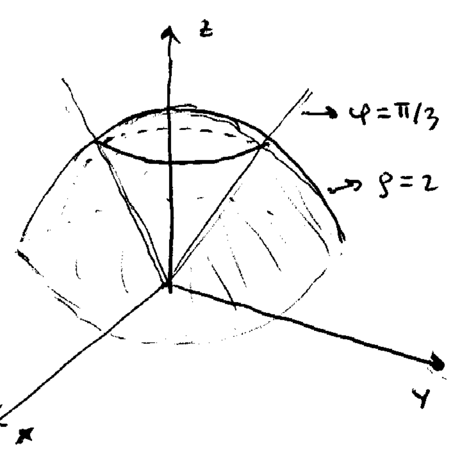
Intersection: $2\sin\vartheta = \sqrt{2} \Rightarrow \vartheta = \frac{\pi}{4}, \frac{3\pi}{4}$ (1pt)

$D = \left\{(r,\vartheta): \sqrt{2} \le r \le 2\sin\vartheta,\ \frac{\pi}{4} \le \vartheta \le \frac{3\pi}{4}\right\}$

$$m = \iint_D dA = \int_{\pi/4}^{3\pi/4}\int_{\sqrt{2}}^{2\sin\vartheta} r\,dr\,d\vartheta = \int_{\pi/4}^{3\pi/4} \frac{r^2}{2}\Big|_{\sqrt{2}}^{2\sin\vartheta} d\vartheta =$$ (1pt)

$$= \int_{\pi/4}^{3\pi/4} (2\sin^2\vartheta - 1)\,d\vartheta = \int_{\pi/4}^{3\pi/4} (-\cos 2\vartheta)\,d\vartheta = -\frac{1}{2}\sin 2\vartheta\Big|_{\pi/4}^{3\pi/4} = \boxed{1}$$ (1pt)

(2) Use spherical coordinates to find the volume of the solid that lies **within** the sphere $x^2+y^2+z^2 = 4$, **above** the xy-plane, and **below** the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$. (Hint: $\tan(\pi/3) = \sqrt{3}$).



$x^2+y^2+z^2 = 4 \iff \rho = 2$ (1pt)

$z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2} \iff \rho\cos\varphi = \frac{1}{\sqrt{3}}\rho\sin\varphi \iff \tan\varphi = \sqrt{3} \iff \varphi = \pi/3$ (1pt)

SOLID $= \left\{(\rho,\vartheta,\varphi): 0 \le \rho \le 2,\ 0 \le \vartheta \le \frac{\pi}{2},\ \frac{\pi}{3} \le \varphi \le \frac{\pi}{2}\right\} = E$

$$V = \iiint_E dV = \int_{\pi/3}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\varphi\,d\rho\,d\vartheta\,d\varphi =$$ (1pt)

$$= 2\pi\int_{\pi/3}^{\pi/2}\sin\varphi\,d\varphi\int_0^2\rho^2\,d\rho =$$

$$= 2\pi\,(-\cos\varphi)\Big|_{\pi/3}^{\pi/2}\ \frac{\rho^3}{3}\Big|_0^2 = 2\pi\,\frac{1}{2}\,\frac{8}{3} = \boxed{\frac{8\pi}{3}\ (= 8.378)}$$ (1pt)