**Math 242 Sec. 9 – Fall 2005** Name: SOLUTION A/B

# FINAL EXAM - A

One page of notes, a hand calculator allowed ~~and your laptop are allowed~~. The test is divided in two parts: a multiple choice part and a traditional problem test part with possible partial credits. **Please, report the solution of the multiple choice part on the included solution page.** For the second part of the test, please, justify your answers and write clearly if you want credit for your work.

(1) [2 Pt.] Find the area of the parallelogram with vertices at $P = (1, 3, -2)$, $Q = (2, 1, 4)$ and $R = (-3, 1, 6)$.

- $(a)$ $A = 1$
- $(b)$ $A = 78$
- $(c)$ $A = \sqrt{1040}$
- $(d)$ $A = \sqrt{1124}$
- [✓ $(e)$ $A = \sqrt{1140}$]
- $(f)$ $A = 82$
- $(g)$ $A = 62$
- $(h)$ $A = 0$

$\vec{PQ} = (1, -2, 6)$

$\vec{QR} = (-5, 0, 2)$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 6 \\ -5 & 0 & 2 \end{vmatrix} = (-4, -32, -10$$

$|\vec{PQ} \times \vec{QR}| = \sqrt{1140}$

(2) [2 Pt.] Find the value(s) $x$ such that the vectors $(3, 2, x)$ and $(2x, 4, x)$ are orthogonal.

- $(a)$ $x = -4$
- $(b)$ $x = -2$
- $(c)$ $x = 0$
- $(d)$ $x = 2$ or $x = 4$
- $(e)$ $x = -2$ or $x = 0$
- [✓ $(f)$ $x = -4$ or $x = -2$]
- $(g)$ $x = -4$ or $x = 0$
- $(h)$ never

$6x + 8 + x^2 = 0$

$\Rightarrow x = -4$ o $x = -2$

(3) [2 Pts.] Find the equation of the plane containing the line: $\mathbf{r}(t) = (1+t, 3-2t, -2+6t)$, and the point $R = (-3, 1, 6)$.

(a) $x - 8y + 3z = -29$
(b) $x + 8y + 3z = 19$
(c) $x - 8y = -22$
(d) $x + 8y = 25$
(e) $2x + 16y - 5z = 60$
(f) $-2x + 16y + 5z = 38$
[✓ (g) $2x + 16y + 5z = 40$]
(h) $2x - 16y + 5z = -56$

$\underline{r}(0) = (1, 3, -2)$
$\underline{r}(1) = (2,$ [illegible] $, 4)$
$R = (-3, 1, 6)$

$2 + 16 \cdot 3$ [illegible] $10 = 40$
$4 + 16 + 20 = 40$
$-6 + 16 + 30 = 40$

(4) [2 Pts.] Find the derivative $\frac{dy}{dx}$, where $y$ is defined by

$$\cos(x-y) = x\,e^y.$$

[✓ (a) $\dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$]

(b) $-\dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$

(c) $\dfrac{\sin(x-y)+e^y}{\sin(x-y)-e^y}$

(d) $-\dfrac{\sin(x-y)+e^y}{\sin(x-y)-e^y}$

(e) $\sin(x-y)-xe^y$

(f) $-\sin(x-y)-xe^y$

(g) $\sin(x-y)$

(h) $-\sin(x-y)$

$$F(x,y) = \cos(x-y) - x e^y$$

$$F_x = -\sin(x-y) - e^y$$

$$F_y = +\sin(x-y) - x e^y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$$

(5) [2 Pts.] Compute the following double integral (Hint: it may be useful to convert to polar coordinates):

$$I = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

(a) $I = \frac{1}{2}\pi$

(b) $I = \pi$

(c) $I = 18\pi$

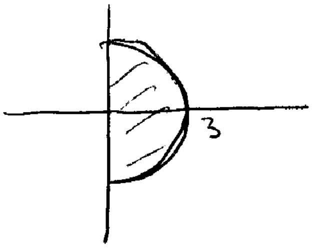
[✓ (d) $I = 9\pi$]

(e) $I = \frac{9}{2}\pi$

(f) $I = \frac{81}{4}\pi$

(g) $I = \frac{81}{8}\pi$

(h) $I = \frac{27}{4}\pi$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r\, r\, dr\, d\vartheta = \pi \left.\frac{r^3}{3}\right|_0^3 =$$

$$= 9\pi$$

(6) [2 Pts.] Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 9$, above the $xy$-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

(a) $V = 27\sqrt{2}\,\pi$
(b) $V = 4\sqrt{2}\,\pi$
(c) $V = 9\sqrt{2}\,\pi/2$
[✓ (d) $V = 9\sqrt{2}\,\pi$]
(e) $V = 8\sqrt{2}\,\pi/3$
(f) $V = 0$
(g) $V = 16\sqrt{2}\,\pi$
(h) $V = 3\pi^2/2$

$$z = \sqrt{x^2+y^2} \iff \varphi = \frac{\pi}{4}$$

$$E = \left\{ (\rho, \vartheta, \varphi) : 0 \le \rho \le 3,\ 0 \le \vartheta \le 2\pi,\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \right\}$$

$$V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 \rho^2 \sin\varphi \, d\rho \, d\vartheta \, d\varphi =$$

$$= 2\pi \left(-\cos\varphi\right)_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left.\frac{\rho^3}{3}\right|_0^3 =$$

$$= \cancel{2}\pi \, \frac{\sqrt{2}}{\cancel{2}} \, 9$$

(7) [2 Pts.] Let $\mathbf{F}(x, y, z) = (4xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + 2(x^2 + z)\,\mathbf{k}$. Find the work $W$ done by the vector field $\mathbf{F}$ in moving a particle along the curve $C$ defined by $x(t) = \sin t, y(t) = \cos t - 1, z(t) = 2\cos t + 1$, for $0 \leq t \leq \pi/2$ (Hint: use the Fundamental Theorem for line integrals).

(*a*) $W = -6$
(*b*) $W = 0$
(*c*) $W = 13$
(*d*) $W = -9$
[✓ (*e*) $W = -5$]
(*f*) $W = 4$
(*g*) $W = 5$
(*h*) $W = -4$

$f(x,y,z) = 2x^2 z + xy^2 + z^2$

$\underline{r}(\tfrac{\pi}{2}) = (1, -1, 1)$

$\underline{r}(0) = (0, 0, 3)$

$f(1,-1,1) = 2 + 1 + 1 = 4$

$f(0,0,3) = 9$

$W = 4 - 9 = -5$

(8) [2 Pts.] Compute the flux $\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$ and $S$ is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane. (Hint: use Divergence Theorem).

(a) $\Phi = 16$
(b) $\Phi = 16\,\pi$
[✓ (c) $\Phi = 32\,\pi$]
(d) $\Phi = \pi/2$
(e) $\Phi = 0$
(f) $\Phi = 8\,\pi$
(g) $\Phi = 64\,\pi$
(h) $\Phi = 5\pi/2$

$$\nabla \cdot \underline{F} = 3x^2 + 3y^2$$

$$\iiint_E (3x^2+3y^2)\,dV =$$

$$= \iint_D (4-r^2)\,3r^2\,dA =$$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\,3r^3\,dr\,d\vartheta$$

$$= 2\pi \int_0^2 (12r^3 - 3r^5)\,dr$$

$$= 2\pi \left(12\frac{r^4}{4} - 3\,\frac{r^6}{6}\right)\Big|_0^2$$

$$= 2\pi \left(3\cdot 2^4 - \frac{2^6}{2}\right)$$

$$= 2\pi\, 2^4(3-2) = \boxed{32\,\pi}$$

(9) [2 Pts.] Evaluate the surface integral $I = \iint_S xyz\, dS$ where $S$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

- [✓ $(a)$ $I = 0$]
- $(b)$ $I = \pi$
- $(c)$ $I = 3\pi/2$
- $(d)$ $I = \pi/2$
- $(e)$ $I = \pi/4$
- $(f)$ $I = 3\pi/4$
- $(g)$ $I = \pi/8$
- $(h)$ $I = \pi/16$

$$\underline{r}(\varphi, \vartheta) = (\sin\varphi \cos\vartheta,\ \sin\varphi \sin\vartheta,\ \cos\varphi)$$

$$|r_\varphi \times r_\vartheta| = \sin\varphi \qquad 0 \le \varphi \le \frac{\pi}{4}$$

$$0 \le \vartheta \le \frac{\pi}{2}$$

$$S = \int_0^{2\pi} \int_0^{\pi/4} \sin^3\varphi \cos\varphi \cos\vartheta \sin\vartheta \, d\varphi \, d\vartheta = 0$$

Since $\int_0^{2\pi} \cos\vartheta \sin\vartheta = 0$

(10) [2 Pts.] Evaluate the following integral (hint: you might have to reverse the order of integration):

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx \quad =$$

- $(a)$ $I = -\frac{1}{12} \cos(1)$
- $(b)$ $I = \frac{1}{12} \cos(1)$
- [✓ $(c)$ $I = \frac{1}{12} (1 - \cos(1))$]
- $(d)$ $I = \frac{1}{12} (\cos(1) - 1)$
- $(e)$ $I = -\frac{1}{4} \cos(1)$
- $(f)$ $I = \frac{1}{4} \cos(1)$
- $(g)$ $I = \frac{1}{4} (1 - \cos(1))$
- $(h)$ $I = \frac{1}{4} (\cos(1) - 1)$

$$= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \sin(y^3) \left. \frac{x^4}{4} \right|_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{4} y^2 \sin(y^3)\, dy = \frac{1}{12} \int_0^1 \sin u\, du$$

$u = y^3$

$du = 3y^2$

$$= \frac{1}{12} (-\cos u) \Big|_0^1$$

$$= \frac{1}{12} (1 - \cos(1))$$