

Solution to September 2007 M⁶ Problem

Let $f(x)$ be a polynomial in one variable with real coefficients, such that the polynomial $f(x) - x$ does not have real roots. Show that the polynomial $f(f(x)) - x$ also does not have real roots.

Solution: From the statement of the problem it follows that the real function $g(x) = f(x) - x$ is continuous and has no zero values. Then by the intermediate value theorem either $f(x) - x > 0$ for all x , or $f(x) - x < 0$ for all x . If $f(x) - x > 0$ for all x , then $f(f(x)) > f(x) > x$ for all x . If $f(x) - x < 0$ for all x then $f(f(x)) < f(x) < x$ for all x . Therefore $f(f(x)) - x$ does not have real roots.