Winning M6 Contest Solution. March 2004

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Problem: Find all solutions of the equation x^2 - 2(y^2) = 1, where x and y are prime numbers.

Solution: x^2 - 1 = 2(y^2), from which it follows that (x-1)(x+1) = 2(y^2). The only prime factors of 2(y^2) are 2 and y, since y itself is prime. Therefore only a few possibilities arise:

Case I: (x-1)=1, (x+1)=2(y^2); therefore, x=2 and x+1 = 2+1 = 3. But 2(y^2) is naturally even and therefore can't equal 3, so no solution comes about with these limitations.

Case II: (x-1)=2(y^2), (x+1)=1; therefore, x=0. But x must be prime, so no solution can result from this case.

Case III: (x-1)=2, (x+1)= y^2; therefore, x=3 and x+1 = 3+1 = 4 =y^2, so y is 2. ONE SOLUTION IS THUS x=3, y=2.

Case IV: (x-1)= y^2, (x+1)=2; therefore, x=1 and x-1 = 1-1 = 0 =y^2, so y is 0. This does not fit the condition that y is prime. No solution possible with this scenario.

Case V: (x-1)=2y, (x+1)=y; therefore, x-1 = 2(x+1), from which it follows that x=-3. x must be a prime number (a negative number is not prime since it has more than just 1 and itself as factors, in fact, it also has in this case -3 and -1 as factors). Therefore no solution possible with this scenario.

Case VI: (x-1)=y, (x+1)=2y; therefore, x+1 = 2(x-1), from which it follows that x=3. Then (x-1) = 3-1 = 2 = y. THE ONE SOLUTION FROM THIS CASE IS THAT x=3, y=2.

Therefore, in all possible cases, it has been shown that the only solution of the above equation with x and y prime, is x=3 and y=2.