# Winning M6 Contest Solution. January 2004

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For what k is the expression k^(k+1) + (k+1)^k divisible by 3?

Solution: The above expression is divisible by 3 whenever k is of the form k=3n+1, where n is an EVEN positive integer or zero. In other words, the above expression is divisible by 3 whenever k is of the form k=6n+1, where n is ANY positive integer or zero. The equivalence of those statements will be shown below. (*Note: since a negative n would produce negative k's, and therefore negative exponents, negative n's are disregarded. The reasoning behind this is because an integer raised to a negative power is always less than one, and therefore can't be divisible by 3.)

Proof: Assume k is an integer. Then k is either of the forms k=3n, k=3n+1 or k=3n+2, where n is a positive integer or zero. The rest of the proof will proceed case-wise.

Case 1: Assume k is of the form k=3n, where n is a positive integer or zero. Then the expression above, by substitution, will yield (3n)^(3n+1) + (3n+1)^(3n). The first part, (3n)^(3n+1) is clearly divisible by 3 since it can be broken up to read (3^(3n+1))(n^(3n+1)). The second part of the expression (3n+1)^(3n) is always equivalent to 1 mod 3. This is because anything 1 mod 3 (in this case, written as 3n+1) multiplied by anything 1 mod 3 (hence, the role of the exponent) will always yield a product of 1 mod 3. With the first half equivalent to 0 mod 3 and the second half equivalent to 1 mod 3, the expression will always be equivalent to 0 + 1 = 1 mod 3. This is never divisible by 3. Therefore, if the expression is divisible by 3, then k is NOT of the form k=3n.

Case 2: Assume k is of the form k=3n+1, where n is a positive integer or zero. Then the expression above, by substitution, will yield (3n+1)^((3n+1)+1) + ((3n+1)+1)^(3n+1). Simplified, this is (3n+1)^(3n+2) + (3n+2)^(3n+1). The first part, as is shown by the argument in case 1, is always equivalent to 1 mod 3, since anything 1 mod 3 (3n+1) multiplied by anything 1 mod 3 (the exponent's role) is always 1 mod 3. The second part, (3n+2)^(3n+1), can either be equivalent to 1 mod 3 or 2 mod 3. The discrepancy here is caused by the fact that if anything 2 mod 3 is multiplied by another 2 mod 3, the result is equivalent to 1 mod 3. However, an odd exponent will leave an "extra" 2 mod 3 (that won't be paired with another 2 mod 3 to yield 1 mod 3). Therefore, anything 2 mod 3 raised to an even exponent will produce an answer equivalent to 1 mod 3, and anything 2 mod 3 raised to an odd exponent will produce an answer equivalent to 2 mod 3.

Recalling the results from evaluating the first half of the expression (where it was determined that the first half is always 1 mod 3), in order to get the whole expression divisible by 3, the second half of the expression needs to be equivalent to 2 mod 3. This occurs when an odd exponent is encountered, and in the case of 3n+1 (the exponent in question), it is odd whenever n is even. Therefore, if k=3n+1, the expression above is divisible by 3 IFF n is an even, positive integer, or zero.

Case 3: Assume that k is of the form k=3n+2, with n being a positive integer or zero. Then the expression above, with the substitution is, (3n+2)^(3n+3) + (3n+3)^(3n+2). It is clear that the second half of the expression is always divisible by 3, since it can be rehashed as (3^(3n+2))((n+1)^(3n+2)). The first half of the expression can either be equivalent to 1 mod 3 or 2 mod 3, depending on the parity of the exponent. Since neither 1 mod 3 + 0 mod 3 nor 2 mod 3 + 0 mod 3 yields a number divisible by 3, the expression can never be divisible by 3. Therefore, if the expression is divisible by 3, then k is NOT of the form k=3n+2.

Conclusion: After reviewing all possible cases of k, it has been proven that the expression given above can only be divisible by 3 if k is of the form k=3n+1, where n is an EVEN, POSITIVE INTEGER, or ZERO. Since n is always even (or equivalent to n=2m, where m is any POSITIVE INTEGER or zero), k=6m+1, where m is any positive integer or zero.