MARCH 2008 M^6 PROBLEM

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There is a unique function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x) + f(y) - 2xy.$$

Proof. We show the unique function is $f(x) = x^2$ by showing

$$f(x-y) = f(x) + f(y) - 2xy \qquad \text{iff} \qquad f(x) = x^2.$$

($\Rightarrow$). Suppose $f(x-y) = f(x) + f(y) - 2xy$. Then $f(0) = 0$ because $f(x) = f(x-0) = f(x) + f(0)$. Thus $f(x-x) = f(0) = 0$. But $f(x-x) = f(x) + f(x) - 2x^2 = 2f(x) - 2x^2$, so $2f(x) = 2x^2$, i.e., $f(x) = x^2$.

($\Leftarrow$). If $f(x) = x^2$, then $f(x-y) = (x-y)^2 = x^2 - 2xy + y^2 = f(x) + f(y) - 2xy$. □

Date: 6 March 2008.