Winning M6 Contest Solution. February 2004

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All values of x must lie on [1, infinity).

Either

Case 1: x^(1/3) > [x/1000],
Case 2: x^(1/3) < [x/1000],
or
Case 3: x^(1/3) = [x/1000]

Case 1

Let x0 = 1000*k for some positive integer k.

Suppose

x^(1/3) > [x/1000] throughout the interval [x0,x0+999].

Then x0^(1/3) > x0/1000 and

x0^(1/3) - x0/1000 > 0.

Let M(x0) = x0^(1/3) - x0/1000
M'(x0) = (1/3)x0^(-2/3) - 1/1000
M''(x0) = (-2/9)x0^(-5/3)

So M'(x0) is always decreasing.

M(31,000) = .41381
M'(31,000) = -.000662)
M(1000) = 9

Since M(1000) > 0,
M(31,000) > 0,
M'(31,000) < 0,
and M''(x0) < 0 for all x0,

M(x0) > 0 for all x0 on [1000, 31,000].

Therefore, for all values of x0 in the interval
[1,31000],

M(x0) > 0 and

x^(1/3) > [x/1000]

Case 2

Again, let x0 = 1000*k for some positive integer k.
Suppose

x^(1/3) < [x/1000] throughout the interval [x0,x0+999].

Then (x0 + 999)^(1/3) < x0/1000 and

(x0 + 999)^(1/3) - x0/1000 < 0

Let N(x0) = (x0 + 999)^(1/3) - x0/1000

N'(x0) = (1/3)(x0 + 999)^(-2/3) - 1/1000
N''(x0) = (-2/9)(x0 + 999)^(-5/3)

N(33,000) = -.6042
N'(33,000) = -.000682

Since N(33,000) < 0,
N'(33,000) < 0,
and N''(x0) < 0 for all values of x0,

N(x0) < 0 for all x0 > 33,000.

Therefore, for all values of x0 in the interval
[33,000,infinity),
N(x0) < 0 and

x^(1/3) < [x/1000]

Since x0 is a multiple of 1000, greater than 31,000
and less than 33,000, x0 must equal 32,000.

So if a solution exists, it must lie on the interval
[32,000 , 32,999], and its cube root must be 32.

32^3 = 32768 does lie on the desired interval.
And since there is only one positive integer whose
cube root is 32, the solution must be unique.

Therefore, the only positive integer satisfying

x^(1/3) = [x/1000] is 32768.