NCSU March 2007 M$^6$ Problem

**Solution to Problem 2**

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**Problem 2.** Is it true that the arithmetic progression with the first term 1 and the difference 729 (1, 730, 1459, ...) has infinitely many terms which are powers of 10?

**Claim:** Yes!

**Proof:** First, assume we have a term in the progression which is a power of 10, i.e., assume there exists integers $n$ and $k$ such that $10^n = 729k + 1$. Taking the square on both sides we have

$$10^{2n} = (729k + 1)^2 = 729m + 1$$

where $m = 729k^2 + 2k$, an integer. Therefore we have discovered a new, larger term in the progression which is a power of 10. Since this can be done infinitely many times, there must be infinitely many terms in the progression that are powers of 10, provided there exists one such term.

Now, for the proof to be complete, we must establish the existence of a single number in the progression that is a power of 10. Mathematically speaking, we must show that $10^n - 1 = 729k$ for some integers $n$ and $k$. This may also be written as $9\sum_{j=0}^{n-1} 10^j = 729k$, or as

$$\sum_{j=0}^{n-1} 10^j = 81k \tag{1}$$

after dividing both sides by 9. Let $\beta = \sum_{j=0}^{80} 10^j$, i.e., $\beta$ is the left-hand side of (1) with $n = 81$. We claim that $\beta$ is divisible by 81. Note that $\beta$ is the number with 81 1's as its only digits. Since the sum of 81 1's is 81, which is divisible by 9, $\beta$ must be divisible by 9. Now, it is easily verified that $111,111,111 \div 9 = 12,345,679$, and therefore

$$\beta \div 9 = 123456790123456790...012345679 \tag{2}$$

where the sequence 12345679 is repeated 9 times, separated by a 0 each time. Therefore, the sum of the digits of (2) is $9 \times 37$, which is obviously divisible by 9. Hence, $\beta$ is divisible by 81. Therefore we have established the existence of $n$ and $k$ for which $10^n = 729k + 1$, and the proof is complete.

**Side Note:** For $n = 81$ the value of $k$ is the following number:

$$1,371,742,112,482,853,223,593,964,334,705,075,445,816,186,556,927,297,668,038,$$

$$408,779,149,519,890,260,631$$