February 2007 M^6 Problem

Solution

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Prove or disprove that the product of 2007 consecutive natural numbers cannot be the 2007th power of a natural number.

Claim: It is **not** possible for the product of 2007 consecutive natural numbers to be the 2007th power of a natural number.

Proof: Suppose x and y are natural numbers for which

$$x(x+1)(x+2)\cdot\ldots\cdot(x+2006) = y^{2007} \tag{1}$$

Clearly, $x < y < x + 2006$. Let p be the largest prime factor of y so that we have $y = p^\alpha d$, where p does not divide d and $\alpha \geq 1$. Then p is also the largest prime factor of the LHS of (1), and consequently, $p \geq 2003 = \max\{\text{primes} \leq 2007\}$.

We therefore see that p can divide at most two factors of the LHS of (1), since these factors must be a distance of at least 2003 apart. Since y is a factor of the LHS of (1), then p must divide either y only (Case 1), or p must divide y and $y + p$ $(= p(p^{\alpha-1}d + 1))$ only (Case 2), or p must divide y and $y - p$ $(= p(p^{\alpha-1}d - 1))$ only (Case 3).

For Case 1, the multiplicity of p on the LHS of (1) is α. For Cases 2 and 3, the multiplicity of p on the LHS is $\alpha + 1$. However, the multiplicity of the RHS is 2007α. In all three cases, we have a contradiction since $\alpha \geq 1$. This completes the proof.