Winning M6 Contest Solution. April 2004

Jason DeVito, jbdevito@unity.ncsu.edu

Submitted on April 05, 2004, at 11:15pm

Prove that min of (a-b)^2, (c-a)^2, and (b-c)^2 is less than
or equal to 1/2 * (a^2 + b^2 + c^2).

Assume that (a-b)^2 is the minimum of the three (if not, relabel
a, b, and c so that it is).

so   (a-b)^2 < (b-c)^2
and  (a-b)^2 < (c-a)^2

so 2* (a-b)^2 < (b-c)^2 + (c-a)^2

Now, we do a proof by contradiction.

Assume that 1/2 * (a^2 + b^2 + c^2) < (a-b)^2

so a^2 + b^2 + c^2 < 2(a-b)^2 < (b-c)^2 + (c-a)^2

so a^2 + b^2 + c^2 < b^2 -2bc + c^2 + c^2 -2ac + a^2

so 0 < -2bc + c^2  - 2ac

or 0 < c^2 + c(-2a-2b).

This is a quadratic in c.

The inequality may only hold if there are no real roots, which
implies that the discriminant should be negative.

So (a+b)^2 < 0, a contradiction

This implies that:

1/2 * (a^2 + b^2 + c^2) < (a-b)^2 is false,

or that:

1/2 * (a^2 + b^2 + c^2) >= (a-b)^2

End Proof

====================================================================

NOTE: Contest coordinator has taken the liberty to correct this solution
slightly, as there may have been problems during the e-mail transmission.