SEPTEMBER 2000 M^6 PROBLEM

Find all continuous functions f on $(0, \infty)$ such that

$$\int_x^{x^2} f(t)\,dt = \int_1^x f(t)\,dt$$

for all $x > 0$.

Winner: The Problem.

Leif Johnson, Alex Schlegel, and David Weaver all found the correct answer to the problem, but no student found a way to justify the answer. Dr. Robert Silber of the Mathematics Department was the only one who solved the problem completely.

Solution:

Using properties of definite integrals, we get

$$\int_x^{x^2} f(t)\,dt = \int_1^{x^2} f(t)\,dt - \int_1^x f(t)\,dt,$$

so by substitution, the condition given in the problem implies that

$$\int_1^{x^2} f(t)\,dt = 2\int_1^x f(t)\,dt.$$

Differentiating both sides of the last equation gives us

$$2xf\left(x^2\right) = 2f(x), \text{ or } f\left(x^2\right) = \frac{f(x)}{x}.$$

For $x > 0$, by applying the last equation repeatedly, we then have

$$f(x) = \frac{f\left(x^{1/2}\right)}{x^{1/2}} = \frac{f\left(x^{1/4}\right)}{x^{1/2}\cdot x^{1/4}} = \cdots = \frac{f\left(x^{1/2^n}\right)}{x^{1/2}\cdot x^{1/4}\cdots x^{1/2^n}}.$$

Since $1/2 + 1/4 + \cdots + 1/2^n = 1 - 1/2^n$, f is a continuous function, and

$$\lim_{n\to\infty} x^{1/2^n} = 1 \text{ and } \lim_{n\to\infty}(1 - 1/2^n) = 1, \text{ we get}$$

$$f(x) = \lim_{n\to\infty} \frac{f\left(x^{1/2^n}\right)}{x^{(1-1/2^n)}} = \frac{f(1)}{x}, \text{ where } f(1) \text{ is any number.}$$