OCTOBER 2000 M^6 PROBLEM

A sequence (a_n) of real numbers is defined by

$$a_1 = 1, a_{n+1} = 1 + a_1 a_2 \cdots a_n \text{ for } n \geq 1.$$

Determine the value of

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

and prove your answer.

WINNER: DAVID JOHNSON

David, a Junior in Mathematics/Computer Science, turned in the only solution to this problem. David noted that $a_{n+1} - 1 = a_1 a_2 \cdots a_n = (a_1 a_2 \cdots a_{n-1}) a_n = (a_n - 1) a_n$ for all $n \geq 2$, and showed by induction that

$$\sum_{i=1}^{n} \frac{1}{a_i} = 2 - \frac{1}{a_{n+1} - 1}, \text{ and } a_{n+1} - 1 \geq n \text{ for all } n.$$

$$\text{Thus } \sum_{n=1}^{\infty} \frac{1}{a_n} = \lim_{n \to \infty} \left(2 - \frac{1}{a_{n+1} - 1} \right) = 2.$$

Alternately, without using induction, you can use David's first observation and partial fractions to show that

$$\frac{1}{a_n} = \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \text{ for } n \geq 2, \text{ so}$$

$$\sum_{i=1}^{n} \frac{1}{a_i} = 1 + \sum_{i=2}^{n} \left(\frac{1}{a_i - 1} - \frac{1}{a_{i+1} - 1} \right) = 2 - \frac{1}{a_{n+1} - 1},$$

because of the telescoping sum.