JANUARY 2001 M^6 PROBLEM

Let $a, b, c, d,$ and e be real numbers. Prove that the roots of

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

cannot *all* be real if $2a^2 < 5b$.

WINNER: THE PROBLEM

No student turned in a solution to this problem. Prof. Bob Silber solved the problem using Rolle's Theorem along with the observation that for a differentiable function f, if $f(x)$ has a zero r with multiplicity n, then $f'(x)$ has the same zero r with multiplicity $n-1$. If we assume that $f(x)$ is the polynomial above and $f(x) = 0$ has 5 real roots, not necessarily all distinct, then $f'(x) = 0$ has 4 real roots, $f''(x) = 0$ has 3 real roots, and

$$f'''(x) = 60x^2 + 24ax + 6b = 0$$

has 2 real roots. But the discriminant of this quadratic is $288(2a^2 - 5b)$, which is negative by hypothesis, so $f(x) = 0$ cannot have all real roots.

This problem can also be done algebraically without calculus, but its tricky.