

Math 132.7

Lesson 7: Population Models and Exponential Functions

7.1 Application

Populations of living animals or plants will have, in general, variable death and birth rates. In many cases we want to make predictions about the population size based on these death and birth rates. Important applications include epidemic control and management of fish populations. The models for these populations can become very complicated. However, one can easily study populations where the death and birth rates are constant. An example is a fruit fly population whose size is small relative to its environment's ability to support that population size.

7.2 Math Model.

Let b be the constant birth rate (number of births per unit time per unit population) and d be the constant death rate (number of deaths per unit time per unit population). We will use r to indicate the overall birth rate; that is, $r=b-d$. If r is positive, then the population will increase; if r is negative, the population will decrease. We want to be more precise about how the population changes. Let the population be given by a function of time, $P(t)$. We wish to form a differential equation for $P(t)$; this will be the math model.

In order to derive the differential equation, we examine how the population will change in some time interval from t to $t + \Delta t$. The change in population should be directly proportional to Δt and the population size:

$$[\text{change in population}] = [\text{births}] - [\text{deaths}]$$

$$P(t + \Delta t) - P(t) \approx \Delta t b P(t) - \Delta t d P(t).$$

In the last line the approximate equality sign was used because the population size varies over this time interval and we used the population size at the beginning of the time interval. We could have used the

population at any time in this interval, but here we will assume the approximation is more accurate for smaller time intervals. Now, divide by Δt , let it go to zero and note the approximation becomes an equality to get

$$\frac{[P(t + \Delta t) - P(t)]}{\Delta t} \approx b P(t) - d P(t)$$

$$P'(t) = b P(t) - d P(t).$$

$$P'(t) = (b - d) P(t)$$

$$P'(t) = r P(t)$$

This simple differential equation is the math model for the population with constant birth and death rates.

If one knows the b , d and the initial population, $P(0)$, then the solution of this model is known to be

$$P(t) = P(0) e^{rt}$$

The number e is approximately 2.71828, and it can be defined to be the number for the base, B , of the exponential function, B^x , such that the slope of the tangent line at $(0, 1)$ is exactly one. This choice of B makes a number of calculations less tedious, and so, e^x is called a "natural" exponential function.

7.3 Method of Solution.

In the calculations below we will consider a number of different differences $b - d$. If the population is initially 10 and increases by 1% per unit time, then

$$P(0) = 10 \text{ and } r = b - d = .01.$$

If the initial population is 20 and decreases by 2% per unit time, then

$$P(0) = 20 \text{ and } r = b - d = -.02.$$

Often the initial population is not known, and several measured populations are recorded. Since there may be measurement errors, one must use the "trendline" for the exponential function to estimate $P(0)$ and $b - d$.

We will illustrate this under the algebra method as we are using the algebraic form of the solution.

7.3.1 Table Method.

t	$10e^{(-.01t)}$	$10e^{(.01)t}$	$10e^{(.02x)}$	$10e^{(.04t)}$
0	10	10	10	10
3	9.70445534	10.3045453	10.6183655	11.2749685
6	9.41764534	10.6183655	11.2749685	12.7124915
9	9.13931185	10.9417428	11.9721736	14.3332941
12	8.86920437	11.2749685	12.7124915	16.160744
15	8.60707976	11.6183424	13.4985881	18.221188
18	8.35270211	11.9721736	14.3332941	20.5443321
21	8.10584246	12.3367806	15.2196156	23.1636698
24	7.86627861	12.7124915	16.160744	26.1169647
27	7.63379494	13.0996445	17.1600686	29.4467955
30	7.40818221	13.4985881	18.221188	33.2011692
33	7.18923733	13.9096813	19.3479233	37.4342138
36	6.97676326	14.3332941	20.5443321	42.2069582
39	6.77056874	14.7698079	21.8147227	47.5882125
42	6.5704682	15.2196156	23.1636698	53.6555597
45	6.37628152	15.6831219	24.5960311	60.4964746
48	6.18783392	16.160744	26.1169647	68.2095847
51	6.00495579	16.6529119	27.7319476	76.906092
54	5.82748252	17.1600686	29.4467955	86.7113766
57	5.65525439	17.6826705	31.2676837	97.7668041
60	5.48811636	18.221188	33.2011692	110.231764
63	5.32591801	18.7761058	35.2542149	124.285967
66	5.16851334	19.3479233	37.4342138	140.132036
69	5.01576069	19.9371553	39.7490163	157.998429
72	4.86752256	20.5443321	42.2069582	178.142732
75	4.72366553	21.1700002	44.8168907	200.855369
78	4.58406011	21.8147227	47.5882125	226.463796
81	4.44858066	22.4790799	50.5309032	255.337217
84	4.31710523	23.1636698	53.6555597	287.891909
87	4.18951549	23.8691085	56.9734342	324.597221
90	4.0656966	24.5960311	60.4964746	365.982344

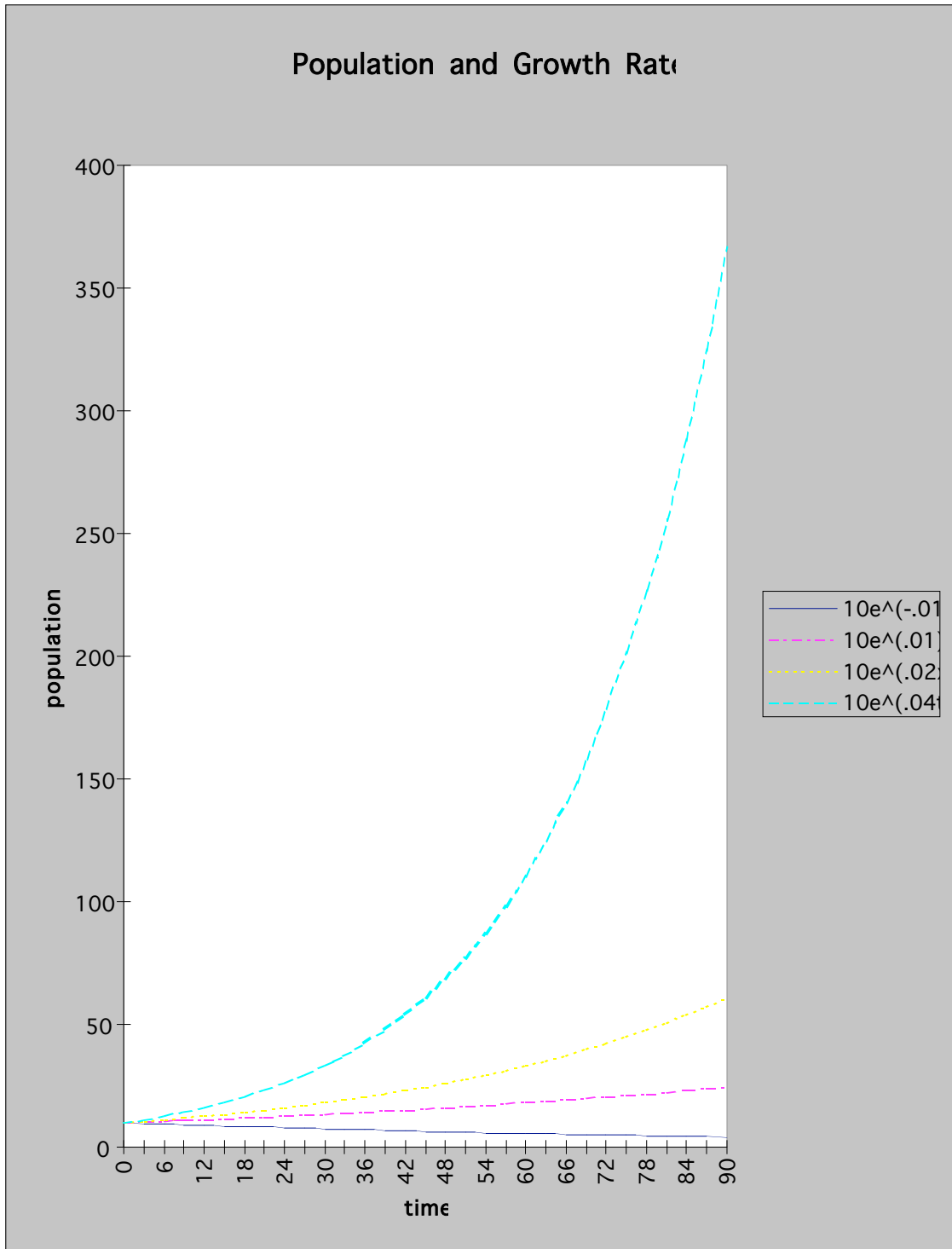
Table: Population with Different b-d

It certainly is tedious looking at all these numbers in this table. The following graph indicates the dramatic effect of doubling b - d. The initial population was 10. After 90 years the population for growth rates of 1%, 2% and 4%, the predicted populations are 25, 60 and 366. These predictions are subject to the growth rate $r = b - d$ remaining constant during the 90 years.

Caveat lector: To enter the formula into Excel, **do not** use the letter e, or a rounded-off approximation to e! Instead, it is best to use for Excel and Maple the command `exp(x)` which represents e^x . For example, to enter the formula $10e^{-0.01t}$ in the table above, if $t=0$ is in cell a2, one should type

=10*exp(-.01*a2)

7.3.2 Graph Method.



7.3.3 Algebra Method.

If we know the model for the population, and want to predict how long it will take the population to grow to a certain size, we can carry out the algebra illustrated in the previous section by hand, or using Maple.

As always, we begin by defining the function that models the population:

```
[> P:=t->623.52*exp(.7323*t);
```

Now we want to know when $P(t)=100000$; to do this, solve the equation in the usual way:

```
[> solve(P(t)=100000);
```

6.933694693

Our model predicts that the population will arrive at 100,000 after about 6.93 units of time.

7.4 Implementation.

Instructions for "Trendline" and Population Data:

- Step 1. Open a spreadsheet and enter the time and population data.
- Step 2. Use the "chart wizard" and form a "scatter" graph of this data.
- Step 3. Form an exponential "trendline" as follows:
 - (a). double click on the data points in the graph,
 - (b). use "add trendlines" under the "chart" menu,
 - (c). select "exponential" function,
 - (d). under the "options" tab choose display equation and forecast forward and backward by 2.

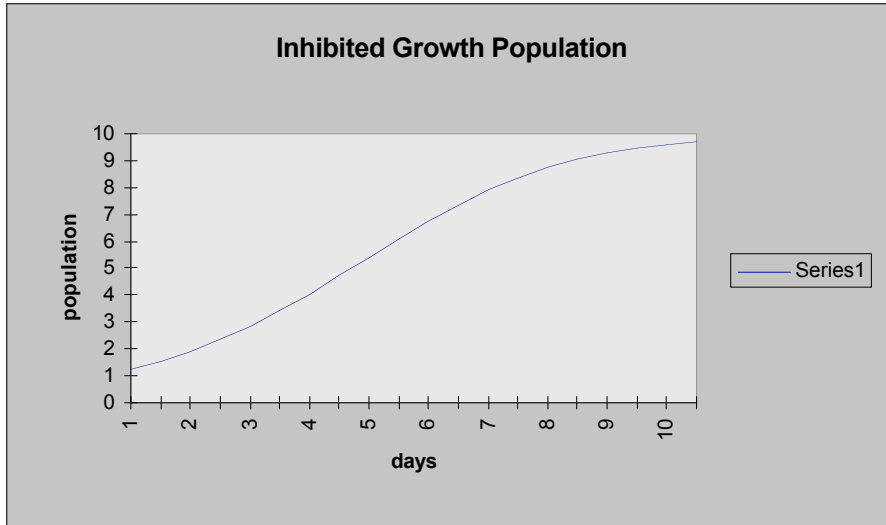
7.5 Assessment.

The population model with constant birth and death rates is not applicable to all populations. If the environment can only support a population up to some M , the maximum population, then the $b - d$ will likely vary with population. If the population is near M , then $b - d$ will be smaller than if the population is near zero. One math model is that $b - d = k(M - P)$ where k is a constant, and this generates the **logistic differential equation**

$$P' = k(M - P)P.$$

Its solution has the function and graph as follows:

$$P(t) = M/(1 + Be^{-kMt}) \text{ and}$$



7.6 Possible Homework.

1. Verify the graphs of $P(t) = P(0) e^{(b-d)t}$ with variable $b - d$.
2. Consider a population whose birth and death rates are constant. Suppose the initial population is 20 and the population is increasing at a rate of 3%.
 - (a). Find the differential equations for its math model.
 - (b). Find the solution and the table and graph forms of the solution.
 - (c). Find the time when the population reaches 40.

3. Consider a fruit fly population with the following data:

Time	Measured Population
1	1,000
2	1,200
3	1,600
4	2,100
5	3,150

Assume the birth and death rates are constants.

- (a). Create a scatter chart of this data.
 - (b). Find an exponential trendline that approximates this data.
 - (c). Use it to find $P(0)$ and $b - d$.
 - (d). When will the population reach 10,000?
4. Consider the data in problem 3. Suppose the measurements for the population had errors of at most plus or minus 10%. How will this effect your solution and predictions?