Math 132.1

Lesson 1: Declining Price, Profits and Graphing

1.1 Application.

A clothing firm can manufacture suits at 20 dollars per suit. The initial cost of the equipment is 10,000 dollars. The revenue was initially 80 dollars per suit, the selling price. But, as more suits were sold, the demand dropped. The retailer "noticed" a trend in the declining price: for every additional suit sold the price had to be dropped by about 5 cents (.05 dollars).

The objective is to find the break-even number of sales where revenue is equal to cost. If the sales are too small, the initial cost will make the total cost larger than the revenue. In the other extreme, if the sales are too large, the price per suit will be very small and the revenue will be too small to realize a profit. Also we would like to know the number of sales where the profit will be a maximum.

1.2 Math Model.

The profit is defined to be the revenue minus the total costs

[profit] = [revenue] - [total cost]

$P(x) = R(x) - C(x)$ where x is the number of suits sold.

The revenue is the price per suit times the number of suits sold, x. The price per suit was initially 80 but declined by .05 dollars for each additional suit sold. This means the price per unit as a function of x, $p(x)$, is a straight line with vertical intercept equal to 80 and slope equal to -.05, that is,

$p(x) = -.05x + 80.$

Thus, the revenue is

[revenue] = [price per suit] times [number of suits]

$R(x) = p(x)x = (-.05x + 80)x.$

The total cost for producing x suits is

[total cost] = [production cost of x suits] + [initial cost]

$$C(x) = 20x + 10{,}000.$$

Therefore, the profit as a function of the number of suits being sold is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-.05x + 80)x - (20x + 10{,}000) \\ &= -.05x^2 + 60x - 10{,}000. \end{aligned}$$

This is a mathematical model of the profit. We may use this model to determine possible break-even level of sales where $P(x) = 0$, and the level of sales where the profit will be a maximum.

1.3 Methods of Solution.

The above profit function is a parabola which must open downward because the coefficient of the squared term is negative. If the total cost is not too large, the parabola will have some parts above the horizontal axis. The break-even sales of suits will be where the parabola intersects the horizontal axis, and the maximum profit is where the parabola peaks. We can find these three points by three methods: (i) calculate table of pairs $(x,P(x))$, (ii) graph the parabola, and (iii) complete the square or use the quadratic formula.

1.3.1 Table Method.

The values of $P(x)$ for a variety of x can be computed by a number of tools. Here we do this via a spreadsheet and for x in increments of 100. Because we know the profit function is a parabola, we can infer from the table that the break-even points are $x = 200$ and $x = 1000$. So, the retailer should configure his/her shop to sell between 200 and 1000 suits. Moreover, from the table we see that the profit is a maximum of 8000 dollars if $x = 600$ suits are sold.

x	P(x)
0	-10000
100	-4500
200	0

300	3500
400	6000
500	7500
600	8000
700	7500
800	6000
900	3500
1000	0
1100	-4500
1200	-10000

Table: Suits and Profit

1.3.2 Graph Method.

The parabolic profit function can easily be graphed. This gives us a visual interpretation of the above table. For large tables of numbers this is much more convenient and may yield additional information. For example, suppose the cost of fabric or cost of labor increased so that the cost of producing one suit rose from 20 dollars to 25 dollars. Then the total cost and the profit are

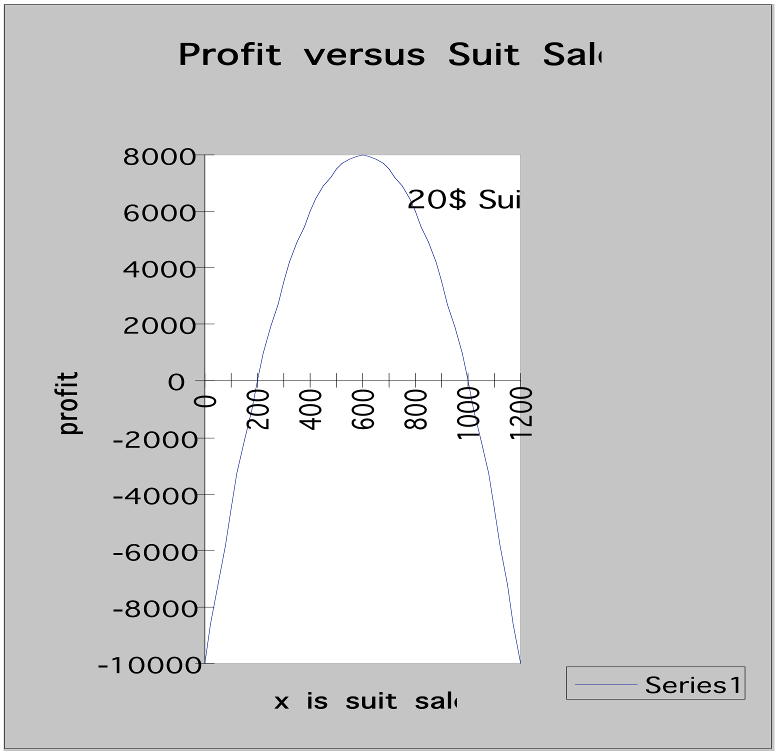
$C(x) = 25x + 10{,}000$ and

$P(x) = R(x) - C(x)$

$= (-.05x + 80)x - (25x + 10{,}000)$

$= -.05x^2 + 55x - 10{,}000.$

Examination of the second graph with the two profit functions indicates that the new break-even points are about x = 220 and x = 880 and have moved closer. The new maximum profit is about $5000 and will be attained when x = 550. What will happen if the price per suit decreases from 20 to 15?


Profit versus Suit Sal
20$ Sui
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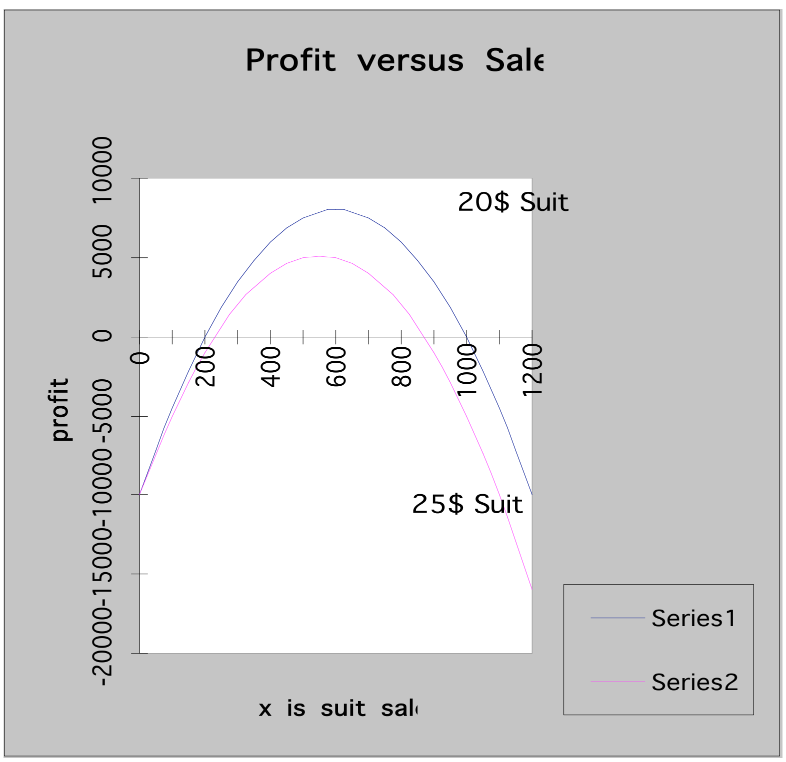
Series1


Profit versus Sale
20$ Suit
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10000
5000
0
-5000
-10000
-15000
-20000
0
200
400
600
800
1000
1200
profit
x is suit sal
Series1
Series2

1.3.3 Algebra Method.

Both the table and graph methods give approximations of the solutions. A more accurate method involves completing the square or using the quadratic formula. Completing the square makes use of the formula

$$a^2 - 2ab + b^2 = (a - b)^2.$$

For the first profit function we have the following calculations

$$P(x) = -.05x^2 + 60x - 10{,}000$$

$$= -1/20\ (x^2 - 1200x + 200{,}000)$$

$$= -1/20\ (x^2 - 2(600)x + (600)^2 - (600)^2 + 200{,}000).$$

Let a = x and b = 600 to get

$$P(x) = -1/20\ ((x - 600)^2 - 160{,}000).$$

The break-even points are found by solving P(x) = 0

$$(x - 600)^2 - 160{,}000 = 0.$$

Thus,

$$x = 600 \pm \sqrt{160{,}000} = 200\, or\, 1000.$$

The maximum profit can be obtained from

$$P(x) = -1/20\ ((x - 600)^2 - 160{,}000).$$

Here P(x) will be the largest possible when $(x - 600)^2$ is the smallest, that is, x = 600. In this case, the maximum profit is P(600) = -1/20 (0 - 160,000) = 8,000. Another more general method is to compute the derivative of the profit function, set it equal to zero and solve for x:

$$P'(x) = (-1/20)2(x - 600)^{2-1}\ 1 = 0 \text{ so that } x = 600.$$

The derivative method is particularly useful if the price function is not linear.

1.3.3 Algebra Method, with Maple

Although taking derivatives of polynomials is not difficult, we can minimize our effort by using a computer algebra system like Maple. Better yet, Maple works with symbols, so it rounds off only when we tell it to do so. This means its solutions are generally more accurate than working with a calculator.

After starting Maple, you should see a blank command prompt:

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[> _
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In this part of the lesson, we will learn how to define functions using Maple. We will learn how to take the derivatives of functions. We will then use the derivative to maximize the profit.

To create a function in Maple, one uses the following format:

[> *name* **:=** *independent_variable* **->** *formula* **;**

First let's discuss the symbols used.

1. The *assignment operator* **:=** tells Maple to associate the rest of the command with the object called *name*. The *name* must start with a letter, but after that first letter you can include numbers and the underscore character (_) if you like.
2. The *functional operator* **->** tells Maple you are defining a function.
3. The *semicolon* **;** tells Maple you have completed a command.

Using this information we can define the profit function for this assignment. We will first define the revenue function *R*, and the cost function *C*. We will then define the profit function *P* as the difference between *R(x)* and *C(x)*:

[>R:=x->(-0.05*x+80)*x;

When you type this into Maple and press the Return key, you will notice that Maple repeats what you've typed, except that the functional operator has now changed to an arrow:

$$R := x \rightarrow (-.05\, x + 80)\, x$$

This is Maple's way of telling you it understood the command. If you get an error instead, look carefully at what you typed, find what you typed wrong, and try again.

Observe that we explicitly wrote the multiplication symbol. This is frequently omitted in mathematical writing, and Maple itself omits it when displaying the function – but, as with Excel, you will need to tell it when to multiply.

Now define the cost function:

[> C:=x->25*x+10000;

Observe that we *did not* type a comma to separate 10 from 000; that would confuse Maple.

Finally define the revenue function. Because some older versions of Maple used on campus do not properly parse functions, we have to determine first what it should be, and only then define it:

[> R(x)-C(x);

$$(-.05\, x + 80)\, x - 25.\, x - 10000.$$

[> P:=x->(-.05*x+80)*x-25*x-10000;

$$P := x \to (-.05\,x + 80)\,x - 25.\,x - 10000.$$

We can now evaluate the value of P at various points to predict the profit at these points. Use the same notation you would use in regular mathematics. To find the value of P when x=10 units have been sold, type:

[> P(10);

$$-9455.$$

So, the company would lose $9,455 if they only sold ten units. Similarly, to find the value of P when x=400 units have been sold, type:

[> P(400);

$$4000.$$

The company would earn $4,000 if they sold four hundred units.

Now let's use Maple to help us maximize the profit. As we discussed earlier, we maximize profit by solving the equation P'(x)=0. In this course we will take derivatives using the D() command. There is a second command, diff(), but we will not study it.

D() accepts one argument, the function whose derivative you want. It returns a function that represents the derivative of the argument.

[> dP:=D(P);

$$dp := x \to -.10\,x + 55$$

Now we have to find the number of items sold that maximize the profit. For this we need to solve the equation P'(x)=0. Since the variable dP represents the function P', we can do this very easily with Maple's solve() function:

[> solve(dP(x)=0);

$$550.$$

Please note: This is the value of x that maximizes the profit; ***it is not the maximum profit!*** To find the maximum profit, substitute this value into the profit function, typing exactly what one would write in algebra or calculus:

[> P(550);

$$5125.00$$

Hence the maximum profit is $5125. It occurs when 550 units are sold.

It may seem difficult, or annoying, to have to learn how to use Maple to do some of these things one could do by hand. However, as the functions become more complicated, Maple will make your life much, much easier.

1.4 Implementation.

Instructions for the Table:

Step 1. Open a spreadsheet and type in x in cell a1 and P(x) in cell b1.

Step 2. In cell a2 type in 0 and in cell b2 type in "=-.05*a2^2+60*a2-10000".

Step 3. In cell a3 type 100 and in cell b3 type in "=-.05*a3^2+60*a3-10000".

Step 4. Select cells a2:b3 and create a "series" by moving the cursor to the lower right corner of cell b3 and "dragging" it down to cell b14.

Step 5. Examine the formulas in the cells b2:b14 and notice that a2 has been incremented by one each move down column b2:b14.

Step 6. You should modify Steps 1-5 to do a table for the second profit function for the 25$ suits.

Instructions for the Graph:

Step 1. Use the "chart wizard" and select the above cells a2:b14.

Step 2. Click the “chart wizard” button.

Step 3. Choose “scatter” chart.

Step 4. Follow the instructions.

Step 5. Experiment with other options, and the table from the profit function for the 25$ suits.

Instructions for Maple:

Step 1. Define the revenue function R and the cost function C.

Step 2. Find the difference R(x)-C(x)

Step 3. Use that result to define the profit function P.

Step 4. Define dP, the derivative of P, using the D() function.

Step 5. Solve for the point of maximum profit using the solve() function.

Step 6. Find the maximum profit by evaluating P at that point.

1.5 Assessment.

The above model has made a number of assumptions. Many of the constants were assumed to be independent of time. We have already discussed the possibility that the cost per suit may vary with time. Also, the initial cost is fixed, but all equipment will eventually need to be repaired or replaced.

The price per suit will in general decrease as the number of suits sold increases. But, why should the price per suit be a linear function? In fact, it is not; any linear function with negative slope and positive y-intercept will eventually intersect the x axis. This means the price of a suit will be zero! For limited x the linear function model for the price per suit may be realistic. In the next lesson we will discuss a method for finding the precise form of the linear function for the price per suit as derived from some past sales data.

Another source of errors comes from computations. When one uses the square root function and a calculating tool such as a calculator or a spreadsheet, the numbers are rounded off to a certain number of digits. Even for the simplest number such as 1/3, the numerical representation is terminated by a finite number of digits, for example, with 8 digits $1/3 \approx .33333333$. In the graphical solution there are only a finite number of pixels or points of light on your computer terminal. So, if a point to be graphed on your screen is between two pixels, then it must be placed in one of the two nearest pixels!

1.6 Possible Homework.

1. Consider the profit function for the 25$ suits

$$P(x) = R(x) - C(x)$$
$$= (-.05x + 80)x - (25x + 10{,}000)$$
$$= -.05x^2 + 55x - 10{,}000.$$

   (a). Find the table form of the solution.

   (b). Find the graph form of the solution.

   (c). Use completing square to find the break-even points and the sales where the profit will be a maximum.

2. Consider the above problem. Assume the same total cost, but suppose the price per suit is believed to be decreasing by .10 dollars every time an additional suit is sold. In this case the price per suit will be -.10x + 80. Repeat the three parts in the first problem.

3. A computer initially sells for $2,000. The price will decrease by $50 for every 100 sold. The initial cost of assembly equipment is $8,000 and the cost for parts and labor for each computer is $1200.

   (a). Find the price per unit as function of x (the number of units sold).

   (b). Find total cost as a function of x.

   (c). Find the profit as a function of x.

   (d). Use the table, graph and completing the square methods to approximate the break-even points and the sales where the profit is a maximum.

4. In problem 3 find the break-even points and maximum profits if the cost per computer varies from $1200, $1100 to $1000.